Introduction

Properties of the inverse trig and hyperbolic functions in Maple depend on Maple's choices of branch cuts and principal branches. Some of this information is available to the user, and some can be deduced. In this month's article, we show how to determine principal branches and branch cuts for these twelve functions, and then provide a tool for assembling the information in a user-friendly graphical format.

The FunctionAdvisor Command

The FunctionAdvisor command provides access to the information Maple has stored for nearly all its special functions. In particular, it can be queried for the branch cuts of a function. For example, applying it to the arctangent function, we find

FunctionAdvisor(branch_cuts, arctan)

[arctan(z), \( z \in \text{ComplexRange} \left( -\infty, -1 \right), z \in \text{ComplexRange} (1, \infty) \)], [arctan(y, x), ComplexRange (\( -\infty, -1 \mid y \)), ComplexRange (1 \mid y, \infty), \text{And}(3(y) = 0, x = 0)]

There are a number of issues to deal with here. First, note that information for both \( \arctan(z) \) and \( \arctan(y, x) \) has been returned. To focus on just the first, issue the command as

FunctionAdvisor(branch_cuts, arctan)[1]

[arctan(z), \( z \in \text{ComplexRange} \left( -\infty, -1 \right), z \in \text{ComplexRange} (1, \infty) \)]

Second, note that the "ComplexRange" construction yields to a conversion to "relation" as in

correct((2.1), relation)
Third, to enter an underscore in Math mode, the escape character (\) should be typed first. Otherwise, Math mode interprets the underscore as a signal to lower the cursor for a subscript.

Finally, note that the "And" construction can be simplified. We show how to do this after we delete \( \arctan(z) \) from with

\[
subsop(1 = \text{NULL}, (2.2))
\]

\[
[\text{And}(\Re(z) = 0, -\infty \leq \Im(z), \Im(z) \leq -1), \text{And}(\Re(z) = 0, 1 \leq \Im(z), \Im(z) \leq \infty)]
\]

Since \( \arctan \) is a list containing two instances of "And", we must map the conversion process onto the list. Thus, we have

\[
\text{map(convert, (2.3), list)}
\]

\[
[[\Re(z) = 0, -\infty \leq \Im(z), \Im(z) \leq -1], [\Re(z) = 0, 1 \leq \Im(z), \Im(z) \leq \infty]]
\]

Table 1 contains the result of all such manipulations for the inverse trig, and inverse hyperbolic functions.

| \[ \arcsin, [z \leq -1, 1 \leq z] \], |
| \[ \arccos, [z \leq -1, 1 \leq z] \], |
| \[ \arctan, [\Re(z) = 0, -\infty \leq \Im(z), \Im(z) \leq -1], [\Re(z) = 0, 1 \leq \Im(z), \Im(z) \leq \infty] \], |
| \[ \text{arccot}, [\Re(z) = 0, -\infty \leq \Im(z), \Im(z) \leq -1], [\Re(z) = 0, 1 \leq \Im(z), \Im(z) \leq \infty] \], |
| \[ \text{arccsc}, [-1 \leq z, z < 0], [0 < z, z \leq 1] \], |
| \[ \text{arcsec}, [-1 \leq z, z < 0], [0 < z, z \leq 1] \], |
| \[ \text{arcsinh}, [\Re(z) = 0, -\infty \leq \Im(z), \Im(z) \leq -1], [\Re(z) = 0, 1 \leq \Im(z), \Im(z) \leq \infty] \], |
| \[ \text{arccosh}, [z < -1, [-1 < z, z \leq 1]] \], |
| \[ \text{arctanh}, [z \leq -1, 1 \leq z] \], |
| \[ \text{arcoth}, [-1 \leq z, z \leq 1] \], |
| \[ \text{arcsech}, [z < 0, 1 \leq z] \], |
| \[ \text{arcsech}, [\Re(z) = 0, -1 \leq \Im(z), \Im(z) < 0], [\Re(z) = 0, 0 < \Im(z), \Im(z) \leq 1]] \] |

| Table 1 | Modified output for \textbf{FunctionAdvisor} applied to each inverse trig and hyperbolic function |
Table 1 was generated as a Maple matrix, using a number of Maple commands to make the desired modifications. The final results are about the best that can be obtained using a basic set of commands. Of course, it would have been possible to typeset (by hand) a more readable version, but instead, we have captured the information in a more visual way, in Figure 4, below.

**Visualizing a Branch Cut**

Figures 1 and 2 are respectively graphs of the real and imaginary parts of the function \( w = \arcsin(z) \), where \( z = x + iy \).

In Figure 2, a discontinuity is apparent along the real axis, and corresponds to the information produced by the `FunctionAdvisor`, namely, that the branch cuts are along \( z \leq -1 \) and \( z \geq 1 \).

**The \textit{branches} Command**

Maple 12 contains an earlier version of the `branches` command, which has been upgraded for the next version of Maple. This command produces a schematic of the principal branches of \( z = f(w) \), where \( f \) is one of the twelve functions in Table 1. For example, Figure 3 displays the...
result of applying the branches command to \( \arcsin(w) \).

This schematic is drawn in the range space, using the notation \( z = \arcsin(w) \), the axis labels being hard-coded with \( \Re(z) \) and \( \Im(z) \). Unfortunately, this notation contradicts the usage in the \texttt{FunctionAdvisor} where the functions are given as \( w = f(z) \).

Fortunately, updated code in the \texttt{branches} command will allow the user to impose alternate labels, and to be consistent with the branch-cut information generated by the \texttt{FunctionAdvisor} command, as we have provided in the composite tool given in Figure 4.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{branches_arcsin.png}
\caption{Application of the \texttt{branches} command to the \arcsin function}
\end{figure}

The magenta boundaries indicate the continuity of the function as the boundary of the principal branch is approached. It is unfortunate that the boundary of the branch to the left of the principal branch uses the color red. It may be difficult to distinguish between these two colors when they are contiguous. Similarly, the branch to the right of the principal branch uses green, and this abuts a line segment drawn in cyan, two nearly identical colors. In the modified version of the \texttt{branches} command, the user has control over the colors used and the labels on the axes. The schematics drawn by the modified code can be seen below in Figure 4.
Compilation of Branch Information

In addition to the information shown in Figures 1, 2, and 3, it is also useful to see the image, under \( w = f(z) \), of the branch cuts of the principal branch. It is also useful to see a graph of the branch cuts themselves - it's a lot easier to comprehend the visual than it is to interpret the analytic information in Table 1.

Figure 4 is a composite of all the information that Maple can provide about principal branches and their branch cuts. On the left, there are graphs of the real and imaginary parts of \( w = f(z) \), where \( f \) is one of the inverse functions selected by clicking on a radio button in the display. In the central column are two graphs drawn in the \( w \)-plane, the upper one generated by the improved branches command; and the lower one being just the image in the \( w \)-plane of the cut under the mapping \( w = f(z) \). The graph on the upper right shows the cut itself in the \( z \)-plane, and this is color-coded to the graphs of the images of the cuts in the \( w \)-plane. This graph is interactive - dragging the "Click and Drag" indicator

with the mouse causes the image of a point in the \( z \)-plane to appear as a red dot in the \( w \)-plane. (To select this indicator, click on the graph and make the selection from the plotting toolbar at the top of the worksheet. Alternatively, apply Context Menu: Manipulator>Click and Drag.)

Finally, at the bottom of Figure 4 the branch cuts are given in interval notation.

<table>
<thead>
<tr>
<th>( w = f(z) ), the Inverse Trig and Hyperbolic Functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \arcsin )</td>
</tr>
</tbody>
</table>
Figure 4  Composite Maple information on the branches and branch cuts of the inverse trig and hyperbolic functions

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