

Physics

Maple provides a state-of-the-art environment for algebraic computations in Physics, with emphasis on ensuring that the computational experience is as natural as possible. The theme of the [Physics project](#) for Maple 2017 has been the consolidation of the functionality introduced in previous releases, together with significant enhancements and new functionality in General Relativity, in connection with classification of solutions to Einstein's equations and tensor representations to work in the embedded 3D curved space, relevant in numerical relativity, and Field Theory, in connection with computational representations for the objects entering the Standard Model in particle physics.

Taking all together, there are more than 300 enhancements throughout the entire package increasing robustness, versatility and functionality, extending once more the range of Physics-related algebraic computations that can be done using computer algebra software and in a natural way.

As part of its commitment to providing the best possible environment for algebraic computations in Physics, Maplesoft launched a [Maple Physics: Research and Development](#) web site with Maple 18, which enabled users to download research versions, ask questions, and provide feedback. The results from this accelerated exchange with people around the world have been incorporated into the Physics package in Maple 2017.

▼ **General Relativity: classification of solutions to Einstein's equations and the Tetrads package**

In Maple 2016, the digitizing of the [database of solutions to Einstein's equations](#) was finished, added to the standard Maple library, with all the metrics from "Stephani, H.; Kramer, D.; MacCallum, M.; Hoenselaers, C.; and Herlt, E., *Exact Solutions to Einstein's Field Equations*". These metrics can be loaded to work with them, or change them, or searched using `g_` (the **Physics** command representing the spacetime metric that also sets the metric to your choice in one go) or using the command [DifferentialGeometry:-Library:-MetricSearch](#).

- Related to these developments, in Maple 2017, the [Physics:-Tetrads](#) package has been vastly improved and extended, now including new commands like [PetrovType](#) and [SegreType](#) to classify these metrics, and the [TransformTetrad](#) now has an option `canonicalform` to automatically derive a transformation and put the tetrad in canonical form (reorientation of the axis of the local system of references), a

relevant step in resolving the equivalence between two metrics.

In the [PDEtools package](#), you have the mathematical tools - including a complete symmetry approach - to work with the underlying partial differential equations. By combining the functionality of the Physics:-Tetrads package, the Physics:-TransformCoordinates command, and the ability to compute Riemann invariants and Weyl scalars, you can also formulate and, depending on the metrics also resolve, the equivalence problem; that is: to answer whether or not, given two metrics, they can be obtained from each other by a transformation of coordinates, as well as compute the transformation.

- Depending on the context, or textbook, two different signatures are used in the description of spacetime: $(- + + +)$, used for instance in the database of solutions of Einstein's equations and related textbook, and $(+ - - -)$ typically used in particle Physics and in Landau's Course for Theoretical Physics, the main reference for the Maple [Physics](#) project. Two other computational representations of those same signatures are $(+ + + -)$ and $(- - - +)$, with time *in position* 4; for the computer, this difference in representation is relevant since it indicates the *computational position* where the component 0 is to be found (either 1 or 4). This difference also changes the position of the time coordinate in the ordered list of the coordinates. Changing the signature to follow a textbook or just for convenience thus entails redefining the metric rearranging lines and columns in its matrix representation and/or changing the sign, and reordering the coordinates, operations prone to mistakes. To handle these redefinitions of the metric and coordinates according to a change in the signature, a new command [Redefine](#) (options [fromsignature](#), [tosignature](#)), got added to the [Physics](#) package.

▼ Examples

▼ Petrov and Segre types, tetrads in canonical form

```
> restart;
with(Physics) :
with(Tetrads);
```

Setting lowercaselatin letters to represent tetrad indices

Defined as tetrad tensors (see ?Physics,tetrads), $e_{a, \mu}, \eta_{a, b}, \gamma_{a, b, c}, \lambda_{a, b, c}$

Defined as spacetime tensors representing the NP null vectors of the tetrad formalism

(see ?Physics,tetrads), $l_{\mu}, n_{\mu}, m_{\mu}, \bar{m}_{\mu}$

```
[IsTetrad, NullTetrad, OrthonormalTetrad, PetrovType, SegreType, TransformTetrad,
e_, eta_, gamma_, l_, lambda_, m_, mb_, n_] (1)
```

There are six Petrov types: I, II, III, D, N and O. Start with a spacetime metric of Petrov type "I" (the numbers always refer to the equation number in the "[Exact solutions to Einstein's field equations](#)" textbook)

> $g_{\mu\nu} = \text{diag}([12, 21, 1])$

Systems of spacetime Coordinates are: $\{X = (t, x, y, \phi)\}$

Default differentiation variables for d_{μ} , D_{ν} and d Alembertian are: $\{X = (t, x, y, \phi)\}$

The McLenaghan, Tariq (1975), Tupper (1976) metric in coordinates $[t, x, y, \phi]$

Parameters: $[a, k, \kappa]$

Comments: k parametrizes the most general electromagnetic invariant with respect to the last 3 Killing vectors

Resetting the signature of spacetime from $"- - - +"$ to $"- + + +"$ in order to match the signature in the database of metrics:

$$g_{\mu, \nu} = \begin{bmatrix} -1 & 0 & 0 & 2y \\ 0 & \frac{a^2}{x^2} & 0 & 0 \\ 0 & 0 & \frac{a^2}{x^2} & 0 \\ 2y & 0 & 0 & x^2 - 4y^2 \end{bmatrix} \quad (2)$$

The Weyl scalars are derived from the Weyl tensor and the null vectors of the Newman-Penrose formalism (l_{μ} , n_{μ} , m_{μ} and \bar{m}_{μ})

> *Weyl[scalarsdefinition]*

$$\Psi_0 = C^{\mu, \nu, \alpha, \beta} l_{\mu} m_{\nu} l_{\alpha} m_{\beta}, \Psi_1 = C^{\mu, \nu, \alpha, \beta} l_{\mu} n_{\nu} l_{\alpha} m_{\beta}, \Psi_2 = C^{\mu, \nu, \alpha, \beta} l_{\mu} m_{\nu} \bar{m}_{\alpha} n_{\beta}, \Psi_3 = C^{\mu, \nu, \alpha, \beta} l_{\mu} n_{\nu} \bar{m}_{\alpha} n_{\beta}, \Psi_4 = C^{\mu, \nu, \alpha, \beta} n_{\mu} \bar{m}_{\nu} n_{\alpha} \bar{m}_{\beta} \quad (3)$$

> *Weyl[scalars]*

$$\Psi_0 = \frac{I|x|}{a^2 x}, \Psi_1 = 0, \Psi_2 = -\frac{1}{a^2}, \Psi_3 = 0, \Psi_4 = \frac{I|x|}{a^2 x} \quad (4)$$

For presentation simplicity, without absolute values, assume that x is positive or 0 and recompute the Weyl scalars

> *Assume($x \geq 0$)*

$$\{x::[0, \infty]\} \quad (5)$$

> *Weyl[scalars]*

$$\Psi_0 = \frac{I}{a^2}, \Psi_1 = 0, \Psi_2 = -\frac{1}{a^2}, \Psi_3 = 0, \Psi_4 = \frac{I}{a^2} \quad (6)$$

Relevant in connection with the equivalence problem between two metrics, the Petrov type of this metric is

> *PetrovType()*

$$"I" \quad (7)$$

and the Plebanski-Segre type is

> *SegreType*();

$$\text{"D", "[(1,1)(11)]"} \quad (8)$$

Two spacetime metrics with different classification cannot be different coordinate representations of the same metric (if the classification is the same the question is still undecided). The same computation but tracking the principal roots behind this classification

> *PetrovType*(*principalroots* = 'ROOTS')

$$\text{"I"} \quad (9)$$

> *ROOTS*

$$\left[\left(\frac{1}{2} + \frac{1}{2} \right) \sqrt{\sqrt{2} \sqrt{5} - 3} \sqrt{2}, \left(-\frac{1}{2} - \frac{1}{2} \right) \sqrt{\sqrt{2} \sqrt{5} - 3} \sqrt{2}, \left(\frac{1}{2} - \frac{1}{2} \right) \sqrt{\sqrt{2} \sqrt{5} + 3} \sqrt{2}, \left(-\frac{1}{2} + \frac{1}{2} \right) \sqrt{\sqrt{2} \sqrt{5} + 3} \sqrt{2} \right] \quad (10)$$

The principal polynomial only, having four different roots corresponding to Petrov type "I"

> *PetrovType*(*principalpolynomial* = *only*)

$$Z^4 + 1 + 6IZ^2 \quad (11)$$

An example of Petrov type III and a canonical form for the corresponding tetrad.

> *g_*[[12, 35, 1]]

Systems of spacetime Coordinates are: {X = (u, x, y, z)}

Default differentiation variables for d_, D_ and dAlembertian are: {X = (u, x, y, z)}

The Kaigorodov (1962), Cahen (1964), Siklos (1981), Ozsvath (1987) metric in coordinates [u, x, y, z]

$$g_{\mu, \nu} = \begin{matrix} \text{Parameters: } [\Lambda] \\ \left[\begin{array}{cccc} 0 & e^{-2z} & 0 & 0 \\ e^{-2z} & e^{4z} & 2e^z & 0 \\ 0 & 2e^z & 2e^{-2z} & 0 \\ 0 & 0 & 0 & \frac{3}{|\Lambda|} \end{array} \right] \end{matrix} \quad (12)$$

For presentation simplicity, assume that z and the parameter Λ are positive

> *Assume*($0 < z, 0 < \Lambda$)

$$\{\Lambda::(0, \infty)\}, \{z::(0, \infty)\} \quad (13)$$

The Petrov and Plebanski-Segre types, the default tetrad and the corresponding Weyl scalars:

> *PetrovType*()

$$\text{"III"} \quad (14)$$

> *SegreType*()

$$\text{"O", "[(1,111)]"} \quad (15)$$

> *e_*[]

$$e_{a, \mu} = \begin{bmatrix} -Ie^{-4z} & -Ie^{2z} & -2Ie^{-z} & 0 \\ Ie^{-4z}\sqrt{2} & 0 & I\sqrt{2}e^{-z} & 0 \\ 0 & 0 & 0 & \frac{\sqrt{3}}{\sqrt{\Lambda}} \\ e^{-4z} & 0 & 0 & 0 \end{bmatrix} \quad (16)$$

> *Weyl*_{scalars}

$$\begin{aligned} \Psi_0 &= \frac{11\Lambda}{4} - 2\Lambda\sqrt{2}, \Psi_1 = \frac{3\Lambda}{4} - \frac{\Lambda\sqrt{2}}{2}, \Psi_2 = \frac{\Lambda}{4}, \Psi_3 = -\frac{3\Lambda}{4} - \frac{\Lambda\sqrt{2}}{2}, \Psi_4 \\ &= \frac{11\Lambda}{4} + 2\Lambda\sqrt{2} \end{aligned} \quad (17)$$

These scalars are not in canonical form. Compute then a canonical form for this tetrad (i.e. rotate the tetrad such that the Weyl scalars match the canonical form shown in the [Petrov classification table](#)) and set this rotated tetrad (using [Setup](#)), then recompute the Weyl scalars

> *Setup*(*e_* = *TransformTetrad*(*canonicalform*)) :

> *Weyl*[*scalars*]

$$\Psi_0 = 0, \Psi_1 = 0, \Psi_2 = 0, \Psi_3 = 1, \Psi_4 = 0 \quad (18)$$

The scalars now match the requirement for canonical form shown in the [Petrov classification table](#) for type III. In this example, the canonical form computed for the tetrad

> *e_*[]

$$\quad (19)$$

$$e_{a, \mu} = \begin{bmatrix} 0 & -\frac{I}{2} \Lambda e^{2z} \sqrt{2} & -\frac{I}{2} \Lambda e^{-z} \sqrt{2} & \frac{\sqrt{\Lambda} \sqrt{3}}{2} \\ \frac{I}{2} e^{-4z} \sqrt{2} & -\frac{13I}{8} e^{2z} \sqrt{2} & -\frac{9I}{8} e^{-z} \sqrt{2} & -\frac{7\sqrt{3}}{8 \Lambda^{3/2}} \\ 0 & \frac{3I}{4} e^{2z} \sqrt{2} & \frac{I}{4} e^{-z} \sqrt{2} & -\frac{\sqrt{3}}{4 \sqrt{\Lambda}} \\ -I e^{-4z} \sqrt{2} & \frac{3I}{4} e^{2z} \sqrt{2} & \frac{I}{4} e^{-z} \sqrt{2} & -\frac{\sqrt{3}}{4 \sqrt{\Lambda}} \end{bmatrix} \quad (19)$$

is not much more complicated than the default tetrad (16) that is not in canonical form, but depending on the example the canonical form can be significantly more complicated.

Examples of transforming the tetrad into canonical form for spacetimes of Petrov types II, N and D can be seen by performing the same steps above departing from the metrics of the database with numbers $g_{[[24, 37, 7]]}$, $g_{[[12, 6, 1]]}$ and $g_{[[12, 8, 4]]}$.

Together with the developments above, the [TransformTetrad](#) got thoroughly reviewed and its transformation functionality extended in order to perform the six traditional rotations that leave different null vectors of the Newman-Penrose formalism unchanged in direction or length,

Load the Schwarzschild metric in spherical coordinates; you can input $g_{[Schwarzschild]}$ or the simpler

> $g_{[sc]}$

Systems of spacetime Coordinates are: $\{X = (t, r, \theta, \phi)\}$

Default differentiation variables for $d_{_}$, $D_{_}$ and $dAlembertian$ are: $\{X = (t, r, \theta, \phi)\}$

The Schwarzschild metric in coordinates $[t, r, \theta, \phi]$

Parameters: $[m]$

$$g_{\mu, \nu} = \begin{bmatrix} \frac{-r + 2m}{r} & 0 & 0 & 0 \\ 0 & -\frac{r}{-r + 2m} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin(\theta)^2 \end{bmatrix} \quad (20)$$

An orthonormal tetrad (this is the one used by default) is

> Setup(tetrad = orthonormal) :

$e_{[\]}$

$$e_{a, \mu} = \begin{bmatrix} \frac{-I\sqrt{-r+2m}}{\sqrt{r}} & 0 & 0 & 0 \\ 0 & \frac{-I\sqrt{r}}{\sqrt{-r+2m}} & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r \sin(\theta) \end{bmatrix} \quad (21)$$

> IsTetrad(%)

Type of tetrad: orthonormal

true

(22)

The key observation here is that the orientation of the axis of the local system of references (the tetrad system) is arbitrary and with them the values of the Weyl scalars too. So, you can perform a Lorentz transformation on the tetrad, without changing the spacetime metric \underline{g} or its Petrov classification while fixing the values of the Weyl scalars in one of the possible forms (classes) shown in the [Petrov classification table](#).

The transformations that you can perform with **TransformTetrad** are either the standard ones, of classes I, II or III specified in the description of [TransformTetrad](#), or arbitrary. For example, transform the orthonormal tetrad above using a null rotation with fixed \underline{l} , where \underline{l} and the related NULL vectors of the Newman-Penrose formalism (commands of the [Tetrads](#) package) satisfy

> $l_{[definition]}$

$$l_{\mu} l^{\mu} = 0, l_{\mu} n^{\mu} = -1, l_{\mu} m^{\mu} = 0, l_{\mu} \bar{m}^{\mu} = 0, g_{\mu, \nu} = -l_{\mu} n_{\nu} - l_{\nu} n_{\mu} + m_{\mu} \bar{m}_{\nu} + m_{\nu} \bar{m}_{\mu} \quad (23)$$

> TransformTetrad(nullrotationwithfixedl_)

$$\begin{bmatrix} \left[\frac{-\frac{1}{2} \sqrt{2} \sqrt{-r+2m}}{\sqrt{r}}, \frac{-\frac{1}{2} \sqrt{2} \sqrt{r}}{\sqrt{-r+2m}}, 0, 0 \right], \\ \left[\frac{-\frac{1}{2} \sqrt{2} \sqrt{-r+2m} (E\bar{E}+1)}{\sqrt{r}}, \frac{-\frac{1}{2} \sqrt{2} \sqrt{r} (E\bar{E}-1)}{\sqrt{-r+2m}}, \frac{\sqrt{2} r (E+\bar{E})}{2}, \right. \\ \left. -\frac{1}{2} \sqrt{2} r \sin(\theta) (E-\bar{E}) \right], \\ \left[\frac{-\frac{1}{2} E \sqrt{2} \sqrt{-r+2m}}{\sqrt{r}}, \frac{-\frac{1}{2} E \sqrt{2} \sqrt{r}}{\sqrt{-r+2m}}, \frac{\sqrt{2} r}{2}, \frac{1}{2} \sqrt{2} r \sin(\theta) \right], \end{bmatrix} \quad (24)$$

$$\left[\frac{-\frac{1}{2} \bar{E} \sqrt{2} \sqrt{-r+2m}}{\sqrt{r}}, \frac{-\frac{1}{2} \bar{E} \sqrt{2} \sqrt{r}}{\sqrt{-r+2m}}, \frac{\sqrt{2} r}{2}, -\frac{1}{2} \sqrt{2} r \sin(\theta) \right]$$

Note the transformation parameter **E** introduced by **TransformTetrad**. You can replace it by a value or [assign](#) it, or you may want to [Assume](#) that this parameter is, for instance, real

> *Assume(E::real)*

$$\{E::real\} \quad (25)$$

> *map(eval, (24))*

$$\left[\left[\frac{-\frac{1}{2} \sqrt{2} \sqrt{-r+2m}}{\sqrt{r}}, \frac{-\frac{1}{2} \sqrt{2} \sqrt{r}}{\sqrt{-r+2m}}, 0, 0 \right], \right. \\ \left[\frac{-\frac{1}{2} \sqrt{2} \sqrt{-r+2m} (E^2+1)}{\sqrt{r}}, \frac{-\frac{1}{2} \sqrt{2} \sqrt{r} (E^2-1)}{\sqrt{-r+2m}}, E \sqrt{2} r, 0 \right], \\ \left[\frac{-\frac{1}{2} E \sqrt{2} \sqrt{-r+2m}}{\sqrt{r}}, \frac{-\frac{1}{2} E \sqrt{2} \sqrt{r}}{\sqrt{-r+2m}}, \frac{\sqrt{2} r}{2}, \frac{1}{2} \sqrt{2} r \sin(\theta) \right], \\ \left. \left[\frac{-\frac{1}{2} E \sqrt{2} \sqrt{-r+2m}}{\sqrt{r}}, \frac{-\frac{1}{2} E \sqrt{2} \sqrt{r}}{\sqrt{-r+2m}}, \frac{\sqrt{2} r}{2}, -\frac{1}{2} \sqrt{2} r \sin(\theta) \right] \right] \quad (26)$$

Now is this a tetrad? Of what type?

> *IsTetrad((26))*

$$\begin{aligned} & \text{Type of tetrad: null} \\ & \text{true} \end{aligned} \quad (27)$$

TransformTetrad applies a transformation on a tetrad but does not set the value of **e_** to this result. For that purpose, i.e. to effectively rotate the tetrad system of references to match the orientation/tetrad you want, use [Setup](#). Following this reorientation of the tetrad, all the components of tensors in the local (tetrad) system of references, as for instance [gamma_](#), [lambda_](#) and the tetrad components of spacetime tensors, automatically change. For example, the Ricci rotation coefficients [gamma_](#) follow this change of value of the components of $e_{a,\mu}$ according to their definition

> *gamma_[definition]*

$$\gamma_{a,b,c} = \left(\mathcal{D}_v(e_{a,\mu}) \right) e_b^\mu e_c^\nu \quad (28)$$

So before setting (26) as the new tetrad, for comparison purposes below, track first the value of the Ricci rotation coefficient with covariant indices 1, 2, 1

> *gamma_[1, 2, 1]*

$$\frac{-Im}{\sqrt{-r+2m} r^{3/2}} \quad (29)$$

Set now (26) as the new tetrad

> Setup(e_ = (26))

$$\left[\begin{aligned} \text{tetrad} &= \left\{ (1, 1) = \frac{-\frac{1}{2} \sqrt{2} \sqrt{-r+2m}}{\sqrt{r}}, (1, 2) = \frac{-\frac{1}{2} \sqrt{2} \sqrt{r}}{\sqrt{-r+2m}}, (2, 1) \right. \\ &= \frac{-\frac{1}{2} \sqrt{2} \sqrt{-r+2m} (E^2 + 1)}{\sqrt{r}}, (2, 2) = \frac{-\frac{1}{2} \sqrt{2} \sqrt{r} (E^2 - 1)}{\sqrt{-r+2m}}, (2, 3) \\ &= E \sqrt{2} r, (3, 1) = \frac{-\frac{1}{2} E \sqrt{2} \sqrt{-r+2m}}{\sqrt{r}}, (3, 2) = \frac{-\frac{1}{2} E \sqrt{2} \sqrt{r}}{\sqrt{-r+2m}}, (3, 3) \\ &= \frac{\sqrt{2} r}{2}, (3, 4) = \frac{1}{2} \sqrt{2} r \sin(\theta), (4, 1) = \frac{-\frac{1}{2} E \sqrt{2} \sqrt{-r+2m}}{\sqrt{r}}, (4, 2) \\ &= \left. \left. \left. \frac{-\frac{1}{2} E \sqrt{2} \sqrt{r}}{\sqrt{-r+2m}}, (4, 3) = \frac{\sqrt{2} r}{2}, (4, 4) = -\frac{1}{2} \sqrt{2} r \sin(\theta) \right\} \right] \end{aligned} \right. \quad (30)$$

The new value of $\gamma_{1,2,1}$

> gamma_[1, 2, 1]

$$\frac{\frac{1}{2} \sqrt{2} m}{\sqrt{-r+2m} r^{3/2}} \quad (31)$$

▼ Equivalence for Schwarzschild metric (spherical and Kruskal coordinates)

▼ Formulation of the problem (remove mixed coordinates)

> restart;

> with(Physics) : with(Tetrads) : Setup(auto = true, mathematicalnotation = true);

Setting lowercaselatin letters to represent tetrad indices

Defined as tetrad tensors (see ?Physics,tetrads), $e_{a,\mu}, \eta_{a,b}, \gamma_{a,b,c}, \lambda_{a,b,c}$

Defined as spacetime tensors representing the NP null vectors of the tetrad

formalism (see ?Physics,tetrads), $l_\mu, n_\mu, m_\mu, \bar{m}_\mu$

** Partial match of 'auto' against keyword 'automaticsimplication'*

[automaticsimplication = true, mathematicalnotation = true]

(32)

The departure point, Schwarzschild metric in spherical coordinates

> $g_{[sc]}$

Systems of spacetime Coordinates are: $\{X = (r, \theta, \phi, t)\}$

Default differentiation variables for d , D and dA lembertian are: $\{X = (r, \theta, \phi, t)\}$

The Schwarzschild metric in coordinates $[r, \theta, \phi, t]$

Parameters: $[m]$

$$g_{\mu, \nu} = \begin{bmatrix} \frac{r}{-r + 2m} & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2 \sin^2(\theta) & 0 \\ 0 & 0 & 0 & \frac{r - 2m}{r} \end{bmatrix} \quad (33)$$

Introduce now Kruskal coordinates following the literature (see Wikipedia) and the corresponding line element involving "mixed" coordinates

> *Coordinates* ($K = [u, \vartheta, \varphi, v]$)

*Systems of spacetime Coordinates are: $\{K = (u, \vartheta, \varphi, v), X = (r, \theta, \phi, t)\}$
 $\{K, X\}$*

(34)

$$ds^2 := \frac{16 (\partial(v)) m^2 e^{-\frac{r}{2m}} (\partial(u)) - \left((1 - \cos(\vartheta))^2 (\partial(\varphi))^2 + (\partial(\vartheta))^2 \right) r^3}{r}$$

$$ds^2 := \frac{16 (\partial(v)) m^2 e^{-\frac{r}{2m}} (\partial(u)) - \left((1 - \cos(\vartheta))^2 (\partial(\varphi))^2 + (\partial(\vartheta))^2 \right) r^3}{r} \quad (35)$$

The mixing of variables is visible: in the line element above is in Kruskal coordinates but you also see r , which belongs to the X (not K) coordinates.

For the purpose of formulating problem *free of this mixing of coordinates*, set the metric now to be (35)

> *Setup*($diff = [K]$, $metric = (35)$, $quiet$) :

> $g_{[]}$

(36)

$$g_{\mu, \nu} = \begin{bmatrix} 0 & 0 & 0 & \frac{8 m^2 e^{-\frac{r}{2m}}}{r} \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2 \sin(\vartheta)^2 & 0 \\ \frac{8 m^2 e^{-\frac{r}{2m}}}{r} & 0 & 0 & 0 \end{bmatrix} \quad (36)$$

To remove the mix of coordinates, introduce a transformation with unknown transformation functions $\{f, h\}$, change variables, and resolve for the transformation functions $\{f, g\}$ (this in itself is resolving a form of *equivalence* problem).

$$> tr_0 := \{u=f(r, t), v=h(r, t)\}$$

$$tr_0 := \{u=f(r, t), v=h(r, t)\} \quad (37)$$

$$> CompactDisplay(tr_0)$$

f(r, t) will now be displayed as f

h(r, t) will now be displayed as h

$$(38)$$

$$> TransformCoordinates(tr_0, g_{[\mu, \nu]})$$

$$\begin{bmatrix} \frac{16 f_r h_r m^2 e^{-\frac{r}{2m}}}{r} & 0 & 0 & \frac{8 m^2 e^{-\frac{r}{2m}} (f_r h_t + h_r f_t)}{r} \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2 \sin(\theta)^2 & 0 \\ \frac{8 m^2 e^{-\frac{r}{2m}} (f_r h_t + h_r f_t)}{r} & 0 & 0 & \frac{16 f_t h_t m^2 e^{-\frac{r}{2m}}}{r} \end{bmatrix} \quad (39)$$

Equate to (33) and solve

$$> convert((39)=rhs((33)), setofequations)$$

$$\left\{ \begin{aligned} 0=0, -r^2 = -r^2, -r^2 \sin(\theta)^2 = -r^2 \sin(\theta)^2, \frac{8 m^2 e^{-\frac{r}{2m}} (f_r h_t + h_r f_t)}{r} = 0, \end{aligned} \right. \quad (40)$$

$$\left. \frac{16 f_r h_r m^2 e^{-\frac{r}{2m}}}{r} = \frac{r}{-r+2m}, \frac{16 f_t h_t m^2 e^{-\frac{r}{2m}}}{r} = \frac{r-2m}{r} \right\}$$

$$> pdsolve((40))$$

$$\left\{ f = _C1 + _C2 \sqrt{r-2m} e^{\frac{r}{4m}} e^{-\frac{t}{4m}}, h = -\frac{e^{\frac{r+t}{4m}} \sqrt{r-2m}}{_C2} + _C3 \right\}, \left\{ f = _C1 \right. \quad (41)$$

$$\left. + _C2 \sqrt{r-2m} e^{\frac{r}{4m}} e^{\frac{t}{4m}}, h = -\frac{\sqrt{r-2m} e^{\frac{r-t}{4m}}}{_C2} + _C3 \right\}$$

Without loss of generality, set $[_C1=0, _C2=1, _C3=0]$

> $tr := combine(subs([_C1=0, _C2=1, _C3=0], eval((37), (41)[1])))$

$$tr := \left\{ u = \sqrt{r-2m} e^{\frac{r-t}{4m}}, v = -e^{\frac{r+t}{4m}} \sqrt{r-2m} \right\} \quad (42)$$

Check it out:

> $g_[]$

$$g_{\mu, \nu} = \begin{bmatrix} 0 & 0 & 0 & \frac{8m^2 e^{-\frac{r}{2m}}}{r} \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2 \sin^2(\vartheta) & 0 \\ \frac{8m^2 e^{-\frac{r}{2m}}}{r} & 0 & 0 & 0 \end{bmatrix} \quad (43)$$

> $TransformCoordinates(tr, g_[\mu, \nu], [X], [K])$

$$\begin{bmatrix} \frac{r}{-r+2m} & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2 \sin^2(\theta) & 0 \\ 0 & 0 & 0 & \frac{r-2m}{r} \end{bmatrix} \quad (44)$$

Compute the inverse of the transformation (42)

> $itr := simplify(normal(solve((42), \{r, t\}), expanded))$

$$itr := \left\{ r = 2 \left(W \left(-\frac{u v e^{-1}}{2m} \right) + 1 \right) m, t = 2 \ln \left(-\frac{v}{u} \right) m \right\} \quad (45)$$

This inverse transformation involves the [LambertW function](#). Set now the metric to be the standard Schwarzschild's metric in spherical coordinates (33) and use itr to get the form of the metric entirely in Kruskal coordinates

> $g_{[sc]}$

Default differentiation variables for d , D and dA lembertian are: $\{X = (r, \theta, \phi, t)\}$

The Schwarzschild metric in coordinates $[r, \theta, \phi, t]$

Parameters: $[m]$

$$g_{\mu, \nu} = \begin{bmatrix} \frac{r}{-r + 2m} & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2 \sin^2(\theta) & 0 \\ 0 & 0 & 0 & \frac{r - 2m}{r} \end{bmatrix} \quad (46)$$

So this is Schwarzschild's solution all in Kruskal coordinates

> $TransformCoordinates(itr, g_{[\mu, \nu]}, [K], [X])$

$$\begin{bmatrix} \left[\begin{array}{c} 0, 0, 0, -\frac{8 W\left(-\frac{u v e^{-1}}{2 m}\right) m^2}{\left(W\left(-\frac{u v e^{-1}}{2 m}\right) + 1\right) u v} \end{array} \right], \\ \left[\begin{array}{c} 0, -4 \left(W\left(-\frac{u v e^{-1}}{2 m}\right) + 1\right)^2 m^2, 0, 0 \end{array} \right], \\ \left[\begin{array}{c} 0, 0, -4 \left(W\left(-\frac{u v e^{-1}}{2 m}\right) + 1\right)^2 m^2 \sin^2(\theta), 0 \end{array} \right], \\ \left[\begin{array}{c} -\frac{8 W\left(-\frac{u v e^{-1}}{2 m}\right) m^2}{\left(W\left(-\frac{u v e^{-1}}{2 m}\right) + 1\right) u v}, 0, 0, 0 \end{array} \right] \end{bmatrix} \quad (47)$$

This metric involves the LambertW function in a non-simplifiable form.

▼ Solving the Equivalence

We now have the two forms: (44) in spherical and (47) in Kruskal coordinates, so we can formulate the equivalence problem from one coordinate system to the other one.

The transformation to be resolved does not need to involve ϕ because neither ϕ nor φ enter either of the two metrics.

The transformation does not need to involve θ or ϑ because they enter the metrics in exactly the same position and with the same dependence.

In addition the Weyl scalars of both metrics are in canonical form and the only scalar different from zero, that is Ψ_2 does not depend on any of $\{\phi, \theta, \varphi, \vartheta\}$

So we look for a generic transformation from spherical to Kruskal of the form

$$\begin{aligned} > \{r=R(K), t=T(K)\} \\ & \qquad \qquad \qquad \{r=R(K), t=T(K)\} \end{aligned} \quad (48)$$

> *CompactDisplay*((48))

$$\begin{aligned} & R(u, \vartheta, \varphi, v) \text{ will now be displayed as } R \\ & T(u, \vartheta, \varphi, v) \text{ will now be displayed as } T \end{aligned} \quad (49)$$

The metric set in this moment is in spherical coordinates, (46), so change using (48) and equate to (47) in Kruskal coordinates

> *convert*(*TransformCoordinates*((48), $g_{[\mu, \nu]}$, $[K]$, $[X]$) = (47), *setofequations*)

$$\left\{ \begin{aligned} \frac{-4 \left(-\frac{R}{2} + m\right)^2 T_u^2 + R_u^2 R^2}{(-R + 2m) R} = 0, & \quad \frac{-4 \left(-\frac{R}{2} + m\right)^2 T_v^2 + R_v^2 R^2}{(-R + 2m) R} = 0, \end{aligned} \right. \quad (50)$$

$$\begin{aligned} & \frac{1}{(-R + 2m) R} \left(-4 \left(-\frac{R}{2} + m\right)^2 T_\varphi^2 + 2 R^2 \left(\frac{R_\varphi^2}{2} + (\cos(\vartheta) + 1) \left(-\frac{R}{2} + m\right) (\cos(\vartheta) - 1) R \right) \right) = -4 \left(W \left(-\frac{u v e^{-1}}{2 m} \right) + 1 \right)^2 m^2 \sin(\vartheta)^2, \\ & \frac{-4 T_v \left(-\frac{R}{2} + m\right)^2 T_u + R_v R_u R^2}{(-R + 2m) R} = -\frac{8 W \left(-\frac{u v e^{-1}}{2 m} \right) m^2}{\left(W \left(-\frac{u v e^{-1}}{2 m} \right) + 1 \right) u v}, \\ & \frac{-4 T_v \left(-\frac{R}{2} + m\right)^2 T_\varphi + R_v R_\varphi R^2}{(-R + 2m) R} = 0, \quad \frac{-4 T_v \left(-\frac{R}{2} + m\right)^2 T_\vartheta + R_v R_\vartheta R^2}{(-R + 2m) R} \\ & = 0, \quad \frac{-4 T_\varphi \left(-\frac{R}{2} + m\right)^2 T_u + R_\varphi R_u R^2}{(-R + 2m) R} = 0, \end{aligned}$$

$$\frac{-4 T_{\varphi} \left(-\frac{R}{2} + m \right)^2 T_{\vartheta} + R_{\varphi} R_{\vartheta} R^2}{(-R + 2m) R} = 0, \frac{-4 T_{\vartheta} \left(-\frac{R}{2} + m \right)^2 T_u + R_{\vartheta} R_u R^2}{(-R + 2m) R}$$

$$= 0, \left. -\frac{R_{\vartheta}^2 R}{R - 2m} - R^2 + \frac{T_{\vartheta}^2 (R - 2m)}{R} = -4 \left(W \left(-\frac{u v e^{-1}}{2m} \right) + 1 \right)^2 m^2 \right\}$$

This is a nonlinear, non-rational PDE system in two unknowns depending on two independent variables (see (48)). You can now either call pdsolve on (50), solving the problem in one step, or first split into cases without solving any differential equation, just doing differential elimination, to see the cases

> PDEtools:-casesplit((50))

$$\left[T_u = \frac{2m}{u}, T_{\vartheta} = 0, T_{\varphi} = 0, T_v = -\frac{2m}{v}, R = 2 \left(\text{LambertW} \left(-\frac{1}{2} \frac{u v e^{-1}}{m} \right) + 1 \right) m \right] \&\text{where} [], \left[T_u = -\frac{2m}{u}, T_{\vartheta} = 0, T_{\varphi} = 0, T_v = \frac{2m}{v}, R = 2 \left(\text{LambertW} \left(-\frac{1}{2} \frac{u v e^{-1}}{m} \right) + 1 \right) m \right] \&\text{where} [] \quad (51)$$

By using differential elimination we removed all nonlinearities, the problem is actually easy for the differential equation routines

> pdsolve((50))

$$\left\{ R = 2 \left(W \left(-\frac{u v e^{-1}}{2m} \right) + 1 \right) m, T = -2m \ln(u) + 2m \ln(v) + _CI \right\}, \left\{ R = 2 \left(W \left(-\frac{u v e^{-1}}{2m} \right) + 1 \right) m, T = 2m \ln(u) - 2m \ln(v) + _CI \right\} \quad (52)$$

So the transformation of coordinates resolving the equivalence between (44) and (47) is

> eval((48), (52))[1]

$$\left\{ r = 2 \left(W \left(-\frac{u v e^{-1}}{2m} \right) + 1 \right) m, t = -2m \ln(u) + 2m \ln(v) + _CI \right\} \quad (53)$$

Check this result transforming (44) fully written in spherical coordinates into (47) fully written in Kruskal coordinates

> g_[]

(54)

$$g_{\mu, \nu} = \begin{bmatrix} \frac{r}{-r+2m} & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2 \sin^2(\theta) & 0 \\ 0 & 0 & 0 & \frac{r-2m}{r} \end{bmatrix} \quad (54)$$

> *TransformCoordinates*((53), $g_{[\mu, \nu]}$, $[K]$, $[X]$)

$$\begin{bmatrix} \left[\left[0, 0, 0, -\frac{8 W\left(-\frac{u v e^{-1}}{2 m}\right) m^2}{\left(W\left(-\frac{u v e^{-1}}{2 m}\right)+1\right) u v} \right], \right. \\ \left. \left[0, -4 \left(W\left(-\frac{u v e^{-1}}{2 m}\right)+1\right)^2 m^2, 0, 0 \right], \right. \\ \left. \left[0, 0, -4 \left(W\left(-\frac{u v e^{-1}}{2 m}\right)+1\right)^2 m^2 \sin^2(\theta), 0 \right], \right. \\ \left. \left. \left[-\frac{8 W\left(-\frac{u v e^{-1}}{2 m}\right) m^2}{\left(W\left(-\frac{u v e^{-1}}{2 m}\right)+1\right) u v}, 0, 0, 0 \right] \right] \right] \end{bmatrix} \quad (55)$$

▼ Redefining the spacetime metric and coordinates according to the signature

> *restart; with(Physics)* :

> *Coordinates(X=cartesian)*

Default differentiation variables for $d_$, $D_$ and dA lembertian are: $\{X = (x, y, z, t)\}$

Systems of spacetime Coordinates are: $\{X = (x, y, z, t)\}$

$\{X\}$ (56)

Consider the generic form of a spacetime metric, for the purpose of illustrating its redefinition (reordering of lines and columns) when changing the signature, it doesn't need to depend on the coordinates

> $G := \text{Matrix}(4, \text{symbol} = g, \text{shape} = \text{symmetric})$

(57)

$$G := \begin{bmatrix} g_{1,1} & g_{1,2} & g_{1,3} & g_{1,4} \\ g_{1,2} & g_{2,2} & g_{2,3} & g_{2,4} \\ g_{1,3} & g_{2,3} & g_{3,3} & g_{3,4} \\ g_{1,4} & g_{2,4} & g_{3,4} & g_{4,4} \end{bmatrix} \quad (57)$$

Set the metric, and in the same call check the value of signature: it is (- - - +)

> *Setup(signature, metric = G)*

$$\begin{aligned} & [metric = \{ (1, 1) = g_{1,1}, (1, 2) = g_{1,2}, (1, 3) = g_{1,3}, (1, 4) = g_{1,4}, (2, 2) = g_{2,2}, (2, 3) \\ & = g_{2,3}, (2, 4) = g_{2,4}, (3, 3) = g_{3,3}, (3, 4) = g_{3,4}, (4, 4) = g_{4,4} \}, signature \\ & = - - - +] \end{aligned} \quad (58)$$

Track the line element

> *g_[lineelement]*

$$\begin{aligned} & g_{3,3} \partial(z)^2 + (2 \partial(x) g_{1,3} + 2 g_{2,3} \partial(y)) \partial(z) + g_{1,1} \partial(x)^2 + 2 g_{1,2} \partial(x) \partial(y) \\ & + g_{2,2} \partial(y)^2 + \partial(t) (\partial(t) g_{4,4} + 2 \partial(x) g_{1,4} + 2 \partial(y) g_{2,4} + 2 \partial(z) g_{3,4}) \end{aligned} \quad (59)$$

Change now the signature by reversing the position of the time-like component, from 4 to 1

> *Setup(signature = "+---")*

$$[signature = + - - -] \quad (60)$$

Neither the metric nor the coordinates change: *t* is still in position 4 and the component (1, 1) of the metric is still g_{11}

> *Coordinates()*

$$\begin{aligned} & \text{Systems of spacetime Coordinates are: } \{X = (x, y, z, t)\} \\ & \{X\} \end{aligned} \quad (61)$$

> *g_[]*

$$g_{\mu, \nu} = \begin{bmatrix} g_{1,1} & g_{1,2} & g_{1,3} & g_{1,4} \\ g_{1,2} & g_{2,2} & g_{2,3} & g_{2,4} \\ g_{1,3} & g_{2,3} & g_{3,3} & g_{3,4} \\ g_{1,4} & g_{2,4} & g_{3,4} & g_{4,4} \end{bmatrix} \quad (62)$$

This design gives you freedom to set things as necessary. However, after changing the signature we may also want to redefine the coordinates - place *t* in position 1 - and possibly also the metric reordering its lines and rows accordingly. A new command for that purpose is *Redefine*, to which you need to indicate the previous signature (or a new signature, to explore the effect of a change before

doing it) and an indication of whether you want to redefine the metric, the coordinates or all:

> *Redefine(coordinates, fromsignature = "---+")*

$$[t, x, y, z] \quad (63)$$

> *Redefine(metric, fromsignature = "---+")*

$$\begin{bmatrix} g_{4,4} & g_{1,4} & g_{2,4} & g_{3,4} \\ g_{1,4} & g_{1,1} & g_{1,2} & g_{1,3} \\ g_{2,4} & g_{1,2} & g_{2,2} & g_{2,3} \\ g_{3,4} & g_{1,3} & g_{2,3} & g_{3,3} \end{bmatrix} \quad (64)$$

These redefinitions however were not set, the keywords *coordinates*, *metric*, or *all* only trigger the change, *t* is still in position 4 and the component (1,1) of the metric is still $g_{1,1}$

> *Coordinates()*

Systems of spacetime Coordinates are: {X = (x, y, z, t)}

$$\{X\} \quad (65)$$

> $g_{\mu, \nu}$

$$g_{\mu, \nu} = \begin{bmatrix} g_{1,1} & g_{1,2} & g_{1,3} & g_{1,4} \\ g_{1,2} & g_{2,2} & g_{2,3} & g_{2,4} \\ g_{1,3} & g_{2,3} & g_{3,3} & g_{3,4} \\ g_{1,4} & g_{2,4} & g_{3,4} & g_{4,4} \end{bmatrix} \quad (66)$$

You can now either use the output of this routine to redefine things using the Setup command, or do all in one go using any of the keywords *setcoordinates*, *setmetric*, or *setall*, for example

> *Redefine(setall, fromsignature = "---+")*

$$[X], \begin{bmatrix} g_{4,4} & g_{1,4} & g_{2,4} & g_{3,4} \\ g_{1,4} & g_{1,1} & g_{1,2} & g_{1,3} \\ g_{2,4} & g_{1,2} & g_{2,2} & g_{2,3} \\ g_{3,4} & g_{1,3} & g_{2,3} & g_{3,3} \end{bmatrix} \quad (67)$$

Now *t* in position 1 and the component (1,1) of the metric equal to $g_{4,4}$

> *Coordinates()*

Systems of spacetime Coordinates are: {X = (t, x, y, z)}

$$(68)$$

{X} (68)

> g_[]

$$g_{\mu, \nu} = \begin{bmatrix} g_{4,4} & g_{1,4} & g_{2,4} & g_{3,4} \\ g_{1,4} & g_{1,1} & g_{1,2} & g_{1,3} \\ g_{2,4} & g_{1,2} & g_{2,2} & g_{2,3} \\ g_{3,4} & g_{1,3} & g_{2,3} & g_{3,3} \end{bmatrix} \quad (69)$$

Note that, despite the reordering of lines and columns in the metric, because we also reordered the variables, the line element has not changed:

> g_[lineelement]

$$g_{2,2} \partial(y)^2 + (2 \partial(t) g_{2,4} + 2 \partial(x) g_{1,2}) \partial(y) + g_{4,4} \partial(t)^2 + 2 g_{1,4} \partial(x) \partial(t) + g_{1,1} \partial(x)^2 + \partial(z) (2 \partial(x) g_{1,3} + 2 g_{2,3} \partial(y) + 2 \partial(t) g_{3,4} + \partial(z) g_{3,3}) \quad (70)$$

> normal((70)-(59))

$$0 \quad (71)$$

This routine is particularly useful when working with metrics from the database of solutions to Einstein's equations, all of which reset the signature to (-+++) when loaded. For example:

> g_[[12, 21, 1]]

Systems of spacetime Coordinates are: {X = (t, x, y, φ)}

Default differentiation variables for d_, D_ and dAlembertian are: {X = (t, x, y, φ)}

The McLenaghan, Tariq (1975), Tupper (1976) metric in coordinates [t, x, y, φ]

Parameters: [a, k, κ0]

Comments: k parametrizes the most general electromagnetic invariant with respect to the last 3 Killing vectors

Resetting the signature of spacetime from "+ - - -" to "- + + +" in order to match the signature in the database of metrics:

$$g_{\mu, \nu} = \begin{bmatrix} -1 & 0 & 0 & 2y \\ 0 & \frac{a^2}{x^2} & 0 & 0 \\ 0 & 0 & \frac{a^2}{x^2} & 0 \\ 2y & 0 & 0 & x^2 - 4y^2 \end{bmatrix} \quad (72)$$

Note above the message about resetting the signature; query about:

> Setup(signature)

[signature = - + + +] (73)

How would this metric and coordinates look with the original signature (---+) ?

> Redefine(all, tosignature = "---+")

$$[x, y, \phi, t], \begin{bmatrix} -\frac{a^2}{x^2} & 0 & 0 & 0 \\ 0 & -\frac{a^2}{x^2} & 0 & 0 \\ 0 & 0 & -x^2 + 4y^2 & -2y \\ 0 & 0 & -2y & 1 \end{bmatrix} \quad (74)$$

By entering the command above replacing *all* by *setall* not only the list of coordinates and metric matrix form are returned but they are also set in one go.

▼ The 3D metric and the ThreePlusOne (3 + 1) new Physics subpackage

In the general theory of relativity, x^1, x^2, x^3 can be any quantities defining the position of bodies in space and the time coordinate x^0 can be defined by an arbitrarily running clock. In order to define simultaneity (synchronizing clocks located at different points in space) as well as determine actual space distances and time intervals in terms of these quantities x^0, x^1, x^2, x^3 , it is relevant to split the spacetime mathematical description of gravity into its space and time parts. For this purpose, three new commands were added to the Physics package:

- [Decompose](#), to decompose 4D tensorial expressions (free and/or contracted indices) into the space and time parts.
- [gamma](#), representing the three-dimensional metric tensor, with which the element of spatial distance is defined as $dl^2 = \gamma_{i,j} dx^i dx^j$.
- [Redefine](#), to redefine the coordinates and the spacetime metric according to changes in the signature from any of the four possible signatures (- + + +), (+ - - -), (+ + + -) and ((- + + +) to any of the other ones.

▼ Examples

> restart; with(Physics) :

> Setup(mathematicalnotation = true)

[mathematicalnotation = true]

(75)

[Define](#) now an arbitrary tensor A

> Define(A)

Defined objects with tensor properties

$$\{A, \gamma_{\mu}, \sigma_{\mu}, \partial_{\mu}, g_{\mu, \nu}, \delta_{\mu, \nu}, \epsilon_{\alpha, \beta, \mu, \nu}\} \quad (76)$$

So A^{μ} is a 4D tensor with only one free index, where the position of the time-like component is the position of the different sign in the signature, that you can query about via

> Setup(signature)

$$[\text{signature} = - - - +] \quad (77)$$

To perform a decomposition into space and time, set - for instance - the lowercase latin letters from i to s to represent spaceindices and

> Setup(spaceindices = lowercase_is)

$$[\text{spaceindices} = \text{lowercaselatin_is}] \quad (78)$$

Accordingly, the 3+1 decomposition of A^{μ} is

> Decompose(A[~mu]);

$$\begin{bmatrix} A^i & A^0 \end{bmatrix} \quad (79)$$

The 3+1 decomposition of the inert representation $\%g_{[\mu, \nu]}$ of the 4D spacetime metric; use the inert representation when you do not want the actual components of the metric appearing in the output

> Decompose(%g_[mu, nu]);

$$\begin{bmatrix} g_{i,j} & g_{i,0} \\ g_{0,j} & g_{0,0} \end{bmatrix} \quad (80)$$

Note the position of the component $\%g_{[0, 0]}$, related to the trailing position of the time-like component in the signature (- - - +).

Compare the decomposition of the 4D inert with the decomposition of the 4D active spacetime metric

> g[]

$$g_{\mu, \nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (81)$$

> Decompose(g_[mu, nu]);

$$\begin{bmatrix} g_{i,j} & 0 \\ 0 & 1 \end{bmatrix} \quad (82)$$

Note that in general the 3D space part of $g_{\mu, \nu}$ is not equal to the 3D metric $\gamma_{i,j}$ whose definition includes another term (see [1] Landau & Lifshitz, eq.(84.7)).

> *gamma_[definition]*

$$\gamma_{i,j} = -g_{i,j} + \frac{g_{0,i}g_{0,j}}{g_{0,0}} \quad (83)$$

The 3D space part of $-g^{\mu, \nu}$ is actually equal to the 3D metric $\gamma^{i,j}$

> *gamma_[~, definition];*

$$\gamma^{i,j} = -g^{i,j} \quad (84)$$

To derive the formula (83) for the covariant components of the 3D metric, **Decompose** into 3+1 the identity

> *%g_[~alpha, ~mu] %g_[mu, beta] = KroneckerDelta[~alpha, beta];*

$$g^{\alpha, \mu} g_{\mu, \beta} = \delta_{\beta}^{\alpha} \quad (85)$$

To the side, for illustration purposes, these are the 3 + 1 decompositions, first excluding the repeated indices, then excluding the free indices

> *Eq := Decompose((85), repeatedindices=false)*

$$Eq := \begin{bmatrix} g^{i, \mu} g_{\mu, j} = \delta_j^i & g^{i, \mu} g_{\mu, 0} = 0 \\ g^{0, \mu} g_{\mu, j} = 0 & g^{0, \mu} g_{\mu, 0} = 1 \end{bmatrix} \quad (86)$$

> *Eq := Decompose((85), freeindices=false)*

$$Eq := g_{0, \beta} g^{\alpha, 0} + g_{i, \beta} g^{\alpha, i} = \delta_{\beta}^{\alpha} \quad (87)$$

Compare with a full decomposition

> *Eq := Decompose((85))*

$$Eq := \begin{bmatrix} g_{0, k} g^{j, 0} + g_{i, k} g^{j, i} = \delta_k^j & g_{0, 0} g^{j, 0} + g_{i, 0} g^{j, i} = 0 \\ g_{0, k} g^{0, 0} + g_{i, k} g^{0, i} = 0 & g_{0, 0} g^{0, 0} + g_{i, 0} g^{0, i} = 1 \end{bmatrix} \quad (88)$$

Eq is a symmetric matrix of equations involving non-contracted occurrences of $g^{0,0}$, $g^{j,0}$ and $g^{j,i}$. Isolate, in *Eq*_{1,2}, $g^{j,0}$, that you input as *%g_[~j, ~0]*, and substitute into *Eq*_{1,1}

> *isolate(Eq[1, 2], %g_[~j, ~0]);*

$$g^{j, 0} = -\frac{g_{i, 0} g^{j, i}}{g_{0, 0}} \quad (89)$$

> subs((89), Eq[1, 1])

$$-\frac{g_{0,k}g_{i,0}g^{j,i}}{g_{0,0}} + g_{i,k}g^{j,i} = \delta_k^j \quad (90)$$

Collect $g^{j,i}$, that you input as %g_[~j, ~i]

> collect((90), %g_[~j, ~i])

$$\left(-\frac{g_{0,k}g_{i,0}}{g_{0,0}} + g_{i,k}\right)g^{j,i} = \delta_k^j \quad (91)$$

Since the right-hand side is the identity matrix and, from (84), $g^{i,j} = -\gamma^{i,j}$, the expression between parenthesis, multiplied by -1, is the reciprocal of the contravariant 3D metric $\gamma^{i,j}$, that is the covariant 3D metric $\gamma_{i,j}$ in accordance to its definition for the signature (- - - +)

> gamma_[definition];

$$\gamma_{i,j} = -g_{i,j} + \frac{g_{0,i}g_{0,j}}{g_{0,0}} \quad (92)$$

▼ References

[1] Landau, L.D., and Lifshitz, E.M. **The Classical Theory of Fields, Course of Theoretical Physics Volume 2**, fourth revised English edition. Elsevier, 1975.

A 3 + 1 description is also key in the study of gravitational waves, black holes, neutron stars and in general to study the evolution of physical system in general relativity by running numerical simulations as traditional initial value (Cauchy) problems. In this framework, Maple 2017 introduces a new package [ThreePlusOne](#), to case Einstein's equations in a 3+1 form, that is, representing spacetime as a stack of nonintersecting 3-hypersurfaces Σ (not necessarily actual space) with:

- Computational representations for the spatial metric $\gamma_{i,j}$ that is induced by $g_{\mu,\nu}$ on the 3-dimensional hypersurfaces, and the related covariant derivative, Christoffel symbols and Ricci and Riemann tensors.
- Computational representations for the *Lapse, Shift, Unit normal and Time vectors* and *Extrinsic curvature* related to the ADM equations.

▼ Examples

> restart; with(Physics) : Setup(coordinatesystems = cartesian);

Default differentiation variables for d_, D_ and dAlembertian are: {X = (x, y, z, t)}

Systems of spacetime Coordinates are: {X = (x, y, z, t)}

[coordinatesystems = {X}]

(93)

> with(ThreePlusOne)

Setting lowercase latin *i* letters to represent space indices

Defined as 4D, spacetime tensors that are purely spatial(see ?Physics,ThreePlusOne),

$$\boldsymbol{\gamma}_{\mu, \nu}, \mathcal{D}_{\mu}, \boldsymbol{\Gamma}_{\mu, \nu, \alpha}, \mathbf{R}_{\mu, \nu}, \mathbf{R}_{\mu, \nu, \alpha, \beta}, \boldsymbol{\beta}_{\mu}, n_{\mu}, t_{\mu}, \mathbf{K}_{\mu, \nu}$$

Changing the signature of spacetime from (- - - +) to (+ + + -) in order to match the signature customarily used in the ADM formalism

[ADMEquations, Christoffel3, D3_, ExtrinsicCurvature, Lapse, Ricci3, Riemann3, Shift, TimeVector, UnitNormalVector, gamma3_] (94)

Note the different color for $\boldsymbol{\gamma}_{\mu, \nu}$, now a 4D tensor representing the metric of a generic 3-dimensional hypersurface (not necessarily the 3D space) induced by the 4D spacetime metric $g_{\mu, \nu}$. All the ThreePlusOne tensors are displayed in black to distinguish them of the corresponding 4D or 3D tensors. The particular hypersurface $\boldsymbol{\gamma}_{\mu, \nu}$ operates is parameterized by the [Lapse](#) α and the [Shift](#) $\boldsymbol{\beta}_{\mu}$.

The induced metric $\boldsymbol{\gamma}_{\mu, \nu}$ is defined in terms of the [UnitNormalVector](#) n_{μ} and the 4D metric $g_{\mu, \nu}$ as

> ThreePlusOne:-gamma3_[definition]

$$\boldsymbol{\gamma}_{\mu, \nu} = n_{\mu} n_{\nu} + g_{\mu, \nu} \quad (95)$$

where n_{μ} is defined in terms of the [Lapse](#) α and the derivative of a scalar function t that can be interpreted as a global time function

> UnitNormalVector[definition]

$$n_{\mu} = -\alpha \mathcal{D}_{\mu}(t) \quad (96)$$

The [TimeVector](#) is defined in terms of the [Lapse](#) α and the [Shift](#) $\boldsymbol{\beta}_{\mu}$ and this vector n_{μ} as

> TimeVector[definition]

$$t_{\mu} = \alpha n_{\mu} + \boldsymbol{\beta}_{\mu} \quad (97)$$

The [ExtrinsicCurvature](#) is defined in terms of the [LieDerivative](#) of $\boldsymbol{\gamma}_{\mu, \nu}$

> ExtrinsicCurvature[definition]

$$\mathbf{K}_{\mu, \nu} = -\frac{\mathcal{L}_n(\boldsymbol{\gamma}_{\mu, \nu})}{2} \quad (98)$$

The metric $\boldsymbol{\gamma}_{\mu, \nu}$ is also a projection tensor in that it projects 4D tensors into the 3D hypersurface Σ . The definitions for the [Christoffel3](#), [Ricci3](#) and [Riemann3](#) tensors, as

well as for any other 4D tensor that is also a 3D tensor in Σ , can thus be written directly by contracting their indices with $\boldsymbol{\gamma}_{\mu, \nu}$. In the case of [Christoffel3](#), [Ricci3](#) and [Riemann3](#), these tensors can be defined by replacing the 4D metric $g_{\mu, \nu}$ by $\boldsymbol{\gamma}_{\mu, \nu}$ and the 4D Christoffel symbols $\Gamma_{\mu, \nu, \alpha}$ by the ThreePlusOne $\mathbf{\Gamma}_{\mu, \nu, \alpha}$ in the definitions of the corresponding 4D tensors. So, for instance

> *Christoffel3[definition]*

$$\mathbf{\Gamma}_{\mu, \nu, \alpha} = \frac{\partial_{\alpha}(\boldsymbol{\gamma}_{\mu, \nu})}{2} + \frac{\partial_{\nu}(\boldsymbol{\gamma}_{\mu, \alpha})}{2} - \frac{\partial_{\mu}(\boldsymbol{\gamma}_{\nu, \alpha})}{2} \quad (99)$$

> *Ricci3[definition]*

$$\mathbf{R}_{\mu, \nu} = \partial_{\alpha}(\mathbf{\Gamma}_{\mu, \nu}^{\alpha}) - \partial_{\nu}(\mathbf{\Gamma}_{\mu, \alpha}^{\alpha}) + \mathbf{\Gamma}_{\mu, \nu}^{\beta} \mathbf{\Gamma}_{\beta, \alpha}^{\alpha} - \mathbf{\Gamma}_{\mu, \alpha}^{\beta} \mathbf{\Gamma}_{\nu, \beta}^{\alpha} \quad (100)$$

> *Riemann3[definition]*

$$\mathbf{R}_{\mu, \nu, \alpha, \beta} = \boldsymbol{\gamma}_{\mu, \lambda} \left(\partial_{\alpha}(\mathbf{\Gamma}_{\nu, \beta}^{\lambda}) - \partial_{\beta}(\mathbf{\Gamma}_{\nu, \alpha}^{\lambda}) + \mathbf{\Gamma}_{\kappa, \alpha}^{\lambda} \mathbf{\Gamma}_{\nu, \beta}^{\kappa} - \mathbf{\Gamma}_{\kappa, \beta}^{\lambda} \mathbf{\Gamma}_{\nu, \alpha}^{\kappa} \right) \quad (101)$$

When working with the ADM formalism, the line element of an arbitrary spacetime metric can be expressed in terms of the differentials of the coordinates dx^{μ} , the [Lapse](#), the [Shift](#) and the *spatial* components of the 3D metric [gamma3_](#). From this line element one can derive the relation between the [Lapse](#), the spatial part of the [Shift](#), the spatial part of the [gamma3_](#) metric and the $g_{0, j}$ components of the 4D spacetime metric.

For this purpose, define a tensor representing the differentials of the coordinates and an [alias](#) $dt = dx^0$

> *Define(dx[mu]);*

Defined objects with tensor properties

$$\left\{ \mathcal{D}_{\mu}, \boldsymbol{\gamma}_{\mu}, \boldsymbol{\sigma}_{\mu}, \mathbf{R}_{\mu, \nu}, \boldsymbol{\beta}_{\mu}, X_{\mu}, \partial_{\mu}, dx_{\mu}, g_{\mu, \nu}, \boldsymbol{\gamma}_{\mu, \nu}, \gamma_{i, j}, \mathbf{\Gamma}_{\mu, \nu, \alpha}, \delta_{\mu, \nu}, \epsilon_{\alpha, \beta, \mu, \nu}, \mathbf{R}_{\mu, \nu, \alpha, \beta}, \boldsymbol{t}_{\mu}, \mathbf{K}_{\mu, \nu}, n_{\mu} \right\} \quad (102)$$

> *alias(dt = dx[~0]) :*

The expression for the line element in terms of the [Lapse](#) and [Shift](#) is (see [2], eq. (2.123))

$$\begin{aligned} > ds^2 = (\text{Shift}[i]^2 - \text{Lapse}^2) dt^2 + 2 \text{Shift}[i] dt dx[\sim i] + \text{gamma}_-[i, j] dx[\sim i] dx[\sim j]; \\ ds^2 = \left(-\boldsymbol{\alpha}^2 + \boldsymbol{\beta}_i \boldsymbol{\beta}^i \right) dt^2 + 2 \boldsymbol{\beta}_i dt dx^i + \gamma_{i, j} dx^i dx^j \end{aligned} \quad (103)$$

Compare this expression with the 3+1 decomposition of the line element in an arbitrary system. To avoid the automatic evaluation of the metric components, work with the inert form of the metric %g_

> $ds^2 = \%g_[\text{mu}, \text{nu}] dx[\sim\text{mu}] dx[\sim\text{nu}];$

$$ds^2 = g_{\mu, \nu} dx^\mu dx^\nu \quad (104)$$

> *Decompose*((104))

$$ds^2 = g_{0,0} dt^2 + g_{0,j} dt dx^j + g_{i,0} dt dx^i + g_{i,j} dx^i dx^j \quad (105)$$

The second and third terms on the right-hand side are equal

> *op*(2, *rhs*((105))) = *op*(3, *rhs*((105)))

$$g_{0,j} dt dx^j = g_{i,0} dt dx^i \quad (106)$$

> *subs*((106), (105))

$$ds^2 = g_{0,0} dt^2 + 2 g_{i,0} dt dx^i + g_{i,j} dx^i dx^j \quad (107)$$

Taking the difference between this expression and the one in terms of the [Lapse](#) and [Shift](#) we get

> *simplify*((107) - (103))

$$0 = (\alpha^2 - \beta_i \beta^i + g_{0,0}) dt^2 + 2 dx^i (g_{i,0} - \beta_i) dt - dx^i dx^j (\gamma_{i,j} - g_{i,j}) \quad (108)$$

Taking coefficients, we get equations for the [Shift](#), the [Lapse](#) and the spatial components of the metric [gamma3_](#)

> $eq_1 := \text{coeff}(\text{coeff}(\text{rhs}((108)), dt), dx[\sim i]) = 0$

$$eq_1 := 2 g_{i,0} - 2 \beta_i = 0 \quad (109)$$

> $eq_2 := \text{coeff}(\text{rhs}((108)), dt^2)$

$$eq_2 := \alpha^2 - \beta_i \beta^i + g_{0,0} \quad (110)$$

> $eq_3 := \text{coeff}(\text{coeff}(\text{rhs}((108)), dx[\sim i]), dx[\sim j]) = 0$

$$eq_3 := -\gamma_{i,j} + g_{i,j} = 0 \quad (111)$$

Using these equations, these quantities can all be expressed in terms of the time and space components of the 4D metric $g_{0,0}$ and $g_{i,j}$

> *isolate*($eq[1]$, *Shift*[i]);

$$\beta_i = g_{i,0} \quad (112)$$

> *isolate*($eq[2]$, *Lapse*²);

$$\alpha^2 = \beta_i \beta^i - g_{0,0} \quad (113)$$

> *isolate*(eq_3 , $\text{gamma_}[i,j]$)

$$\gamma_{i,j} = g_{i,j} \quad (114)$$

▼ References

- [1] Landau, L.D., and Lifshitz, E.M. **The Classical Theory of Fields, Course of Theoretical Physics Volume 2**, fourth revised English edition. Elsevier, 1975.
- [2] Baumgarte, T.W., Shapiro, S.L., **Numerical Relativity, Solving Einstein's Equations on a Computer**, Cambridge University Press, 2010.

▼ Tensors in Special and General Relativity

A number of relevant changes happened in the tensor routines of the [Physics](#) package, towards making the routines pack more functionality both for special and general relativity, as well as working more efficiently and naturally, based on Maple's Physics users' feedback collected during 2016.

▼ New functionality

- New command `EnergyMomentum`, to represent the Energy-Momentum tensor
- Implement conversions to most of the tensors of general relativity (relevant in connection with functional differentiation)
- New setting in the Physics Setup allows for specifying the [cosmologicalconstant](#) and a default [tensorsimplifier](#)

▼ The StandardModel new Physics subpackage

One of the interesting things about the [Physics package](#) is that it was designed from scratch to extend the domain of operations of the Maple system *from commutative variables* to one that includes *commutative, anticommutative and noncommutative variables*, as well as symbolic tensors, abstract vectors and related (nabla) differential operators. Using this framework, Maple 2017 introduces a computational representation for the Standard Model of particle physics. The Standard Model is the $SU_c(3) \times SU_L(2) \times U(1)$ gauge theory describing the electromagnetic, weak, and strong interactions, as well as classifying all known elementary particles. This new StandardModel subpackage includes computational representations for:

- Field-function representations for fermions (quarks and leptons), gauge bosons, and the Higgs boson
- Gell-Man matrices
- Covariant derivative gauge terms integrated into the [D_](#) covariant derivative command formerly used only for general relativity