Computation of the $\mathcal{L}_\infty$-norm of finite-dimensional linear systems

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- Let A, B, C, D be real matrices of respective orders $n \times n$, $n \times m$, $p \times n$, $p \times m$.
- *Linear dynamical system*:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

  - x: *the state vector*,
  - y: *the output vector*,
  - u: *the control vector*.

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$$G(s) = C(sI_q - A)^{-1}B + D$$

Definition

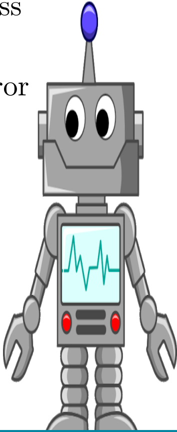
- $\mathcal{R}H_\infty = \{\frac{a(s)}{b(s)} \in \mathbb{R}(s) | \deg(a) \leq \deg(b), b(s) = 0 \Rightarrow \mathrm{Re}(s) < 0\}$,
- $\mathcal{R}L_\infty = \{\frac{a(s)}{b(s)} \in \mathbb{R}(s) | \deg(a) \leq \deg(b), b(i\omega) \neq 0 \ \forall \omega \in \mathbb{R}\}$,
- For $G \in \mathcal{R}L_\infty^{u\times v} = \mathcal{R}\mathcal{L}_\infty$,

$$||G||_\infty := \operatorname*{ess\,sup}_{\omega\in\mathbb{R}} \bar{\sigma}\{G(i\omega)\}$$

- $\bar{\sigma}$: The maximal singular value of a matrix.
- $\bar{\lambda}$: The maximal eigenvalue of a square matrix.
- M^*: The Hermitian transpose of M.

$$\boxed{\bar{\sigma}(M) = \bar{\lambda}(M^* M)^{1/2}}$$

- In robust control, $||G||_\infty$ is considered as a robustness measure.
- In model order reduction, $||G||_\infty$ is a tool for the error measurement [Zhou et al.(1996)Zhou, C., and Glover]



- Let $\gamma > 0$, $G \in \mathcal{RL}_\infty$ and $\Phi_\gamma(s) := \gamma^2 I - G^*(s)G(s)$.
- For $G \in \mathcal{RL}_\infty$, $G^*(s) = G^T(-s)$.
- $\det(\Phi_\gamma(s)) \in \mathbb{Q}[\gamma^2](s^2)$, $s = i\omega$:

$$\det(\Phi_\gamma(s)) = \frac{\boxed{n(\gamma,\omega)}}{d(\omega)}$$

.

Properties

- τ_P: coefficients bitsize of $P \in \mathbb{R}[x]$.

Proposition

$G = (\frac{P_{i,j}}{Q_{i,j}})_{i,j} \in \mathcal{R}L_{\infty}^{u\times v}$, $n(\gamma,\omega) = \texttt{numer}(\det(\Phi_\gamma(s)))$.

- $\alpha = \max(u, v)$,
- $N = max(deg_s(Q_{i,j}))$,
- $\tau_G = max\{\tau_{P_{i,j}}, \tau_{Q_{i,j}},\ i,j \in \{1,\dots,\alpha\}\}$.

$$\boxed{\deg_\gamma(n) = \mathcal{O}(\alpha),\ \deg_\omega(n) = \mathcal{O}(\alpha^2 N)\ , \tau_n = \tilde{\mathcal{O}}(\alpha^2\tau_G).}$$

Properties

Theorem

- $G \in \mathcal{RL}_\infty$, $n(\gamma,\omega) = \texttt{numer}(\det(\Phi_\gamma(s)))$:

$$\max(\pi_\gamma(V_{\mathbb{R}^2}(n))) = ||G||_\infty < \infty$$

$G(s) = \frac{a(s)}{b(s)} \in \mathcal{R}L_\infty$:

- $||G||_\infty = \mathsf{sup}_{\gamma \in \mathbb{R}_{\geq 0}} \left\{ \exists\, \omega \in \mathbb{R} \mid \gamma^2 = |G(i\omega)|^2 \right\}$.
- $|G(i\omega)|^2 \in \mathbb{Q}(\omega^2)$.

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$\|G\|_\infty$:

1. $Z := V_{\mathbb{R}}(\texttt{numer}(\frac{\partial |G(i\omega)|^2}{\partial \omega}))$.
2. $\bar{\gamma} = \max\{|G(i\omega_k)|,\ \omega_k \in Z\}$.
3. $\|G\|_\infty = \max\{\bar{\gamma},\ |G(i\infty)|\ \}$.

- $Lc_x(P)$: leading coefficient of $P \in \mathbb{R}[x, y]$ w.r.t x.
- $\pi_\gamma : (\gamma, \omega) \mapsto \gamma$.

The problem: $G = (\frac{P_{i,j}}{Q_{i,j}})_{i,j} \in \mathcal{R}L_\infty^{u \times v}$:

$$||G||_\infty = \max\left\{\pi_\gamma\left(V_{\mathbb{R}^2}\left(\bar{n}, \frac{\partial \bar{n}}{\partial \omega}\right)\right) \cup V_{\mathbb{R}}\left(Lc_\omega(\bar{n})\right)\right\}$$

Existing methods

- The standard methods for the $\mathcal{L}_\infty$-norm computation are numerical: e.g., bisection algorithms, eigenvalues computation of Hamiltonian matrices [Boyd et al.(1989)Boyd, Balakrishnan, and Kabamba, Chen et al.(1990)Chen, Maza, and Xie].
- In their paper, [Kanno and Smith(2006)] develop a validated numerical algorithm for the $\mathcal{L}_\infty$-norm computation.
- [Chen et al.(2013)Chen, Mazza, and Xie] provide an equivalent study using the theory of border polynomials, which makes the presentation of their solution simpler:
  - The $\|.\|_\infty$ problem is reduced to the computation of the maximal x-projection of the real solutions (x, y) of a bivariate polynomial system $\{\mathcal{P}, \frac{\partial \mathcal{P}}{\partial y}\} \subset \mathbb{Z}[x, y]$.

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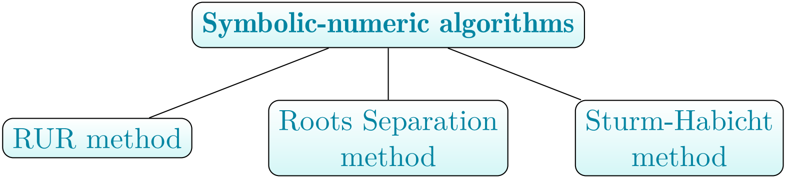
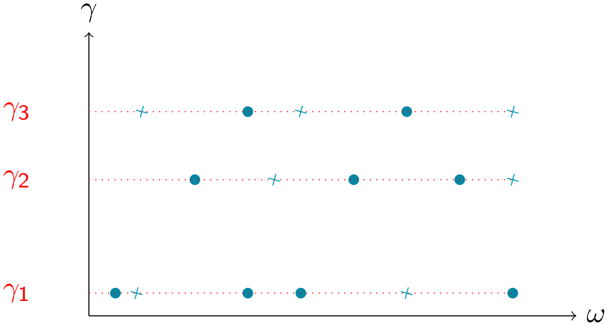
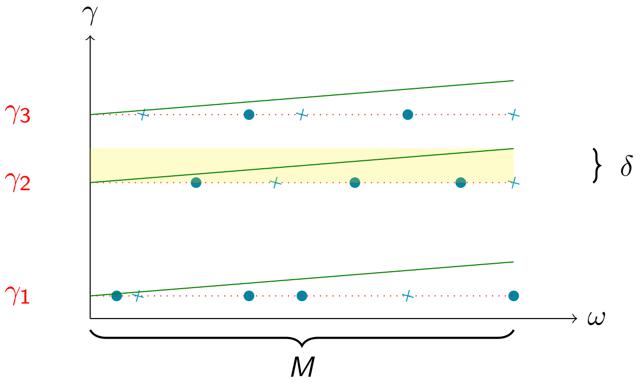

Symbolic-numeric algorithms
RUR method
Roots Separation method
Sturm-Habicht method

- ⋆ Zero-dimensional $\Sigma = \{n(\gamma,\omega), \frac{\partial n}{\partial \omega}(\gamma,\omega)\} \subset \mathbb{Z}[\gamma,\omega]$,
- ⋆ $\deg(n) < d$.

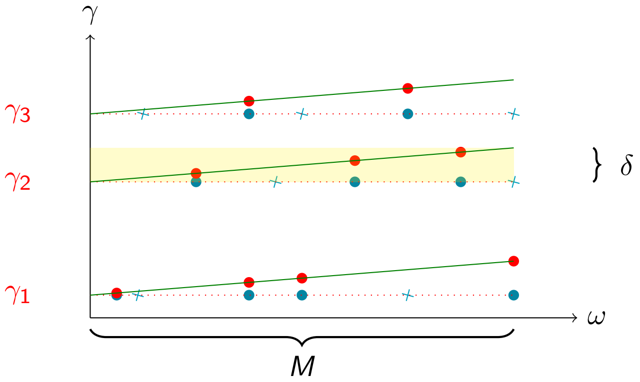
1. Computing linear separating form $t = \gamma + a.\omega$:
   - Bitsize of a: $\mathcal{O}(\log\ d)$.
   - $\Psi_a(\Sigma) \subset \mathbb{Z}[\gamma + a\omega, \omega]$, $p = \mathtt{Res}(\Psi_a(\Sigma), \omega)$.
2. RUR decomposition (such as a triangular decomposition):
   - $\begin{cases} \gamma = \frac{p_0(t)}{q(t)} \\ \omega = \frac{p_1(t)}{q(t)} \end{cases}$
   - $\xi \in V_{\mathbb{R}}(p) \implies (\gamma,\omega) = \left(\frac{p_0(\xi)}{q(\xi)}, \frac{p_1(\xi)}{q(\xi)}\right) \in V_{\mathbb{R}}(\Sigma)$.
3. Finding the isolating boxes of the solutions.



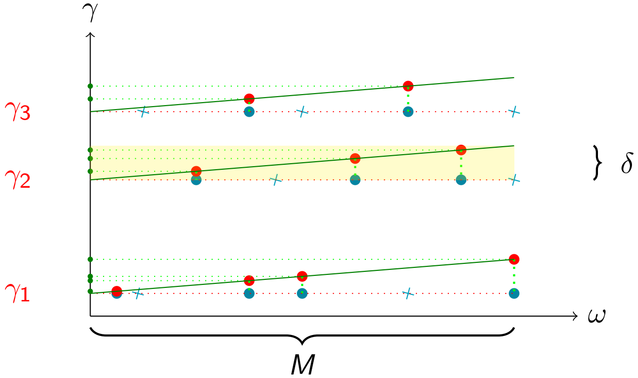
- $R_\gamma = \mathtt{Res}(\Sigma, \omega)$.
- $\{\gamma_1, \gamma_2, \gamma_3\} = V_{\mathbb{R}}(R_\gamma)$.



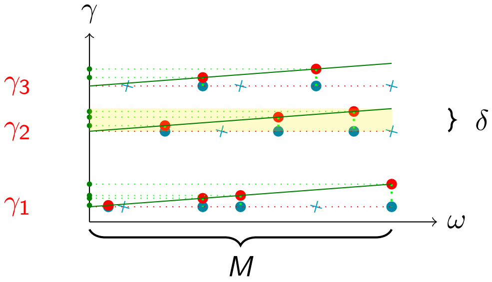
- $\delta < \frac{1}{2}\min\{|\gamma_{i+1} - \gamma_i|, \ \forall \gamma_i \in \pi_\gamma(V(n, \frac{\partial n}{\partial \omega}))\}$,
- $M > \max\{\omega, \forall(\gamma, \omega) \in V(n, \frac{\partial n}{\partial \omega})\}$.
- $0 < s < \frac{\delta}{M}$.
- Bitsize of s: $\mathcal{O}(d^3\tau)$.



- $\Psi_s(\Sigma) \subset \mathbb{Z}[\gamma + s\omega, \omega]$.



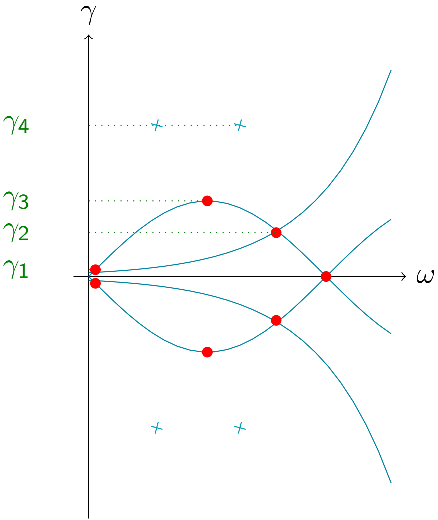
- $R_T = \mathtt{Res}(\Psi_s(\Sigma), \omega)$.



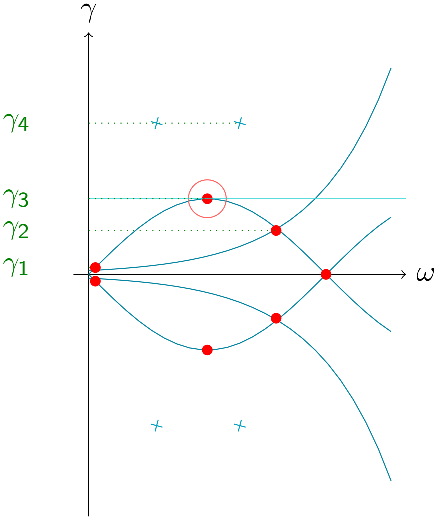
Proposition

- $\gamma_m \in V_{\mathbb{R}}(R_\gamma)$,
- $\gamma'_{max} = \max\{V_{\mathbb{R}}(R_T)\}$, *where* $R_T = \mathtt{Res}(\Psi_s(\Sigma), \omega)$.

If $\gamma'_{max} \in [\gamma_m - \delta, \gamma_m + \delta]$ *then* $\gamma_m = \max\{\pi_\gamma(V_{\mathbb{R}^2}(\mathcal{P}, \mathcal{Q}))\}$.



- $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\} = V_{\mathbb{R}}(R_\gamma)$.



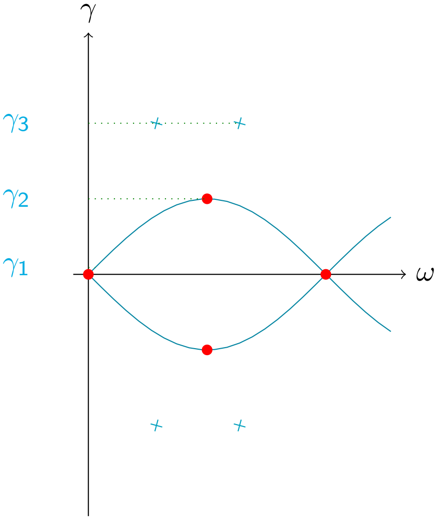
$\gcd(n(\gamma_3, \omega), \frac{\partial n}{\partial \omega}(\gamma_3, \omega))$

Lemma

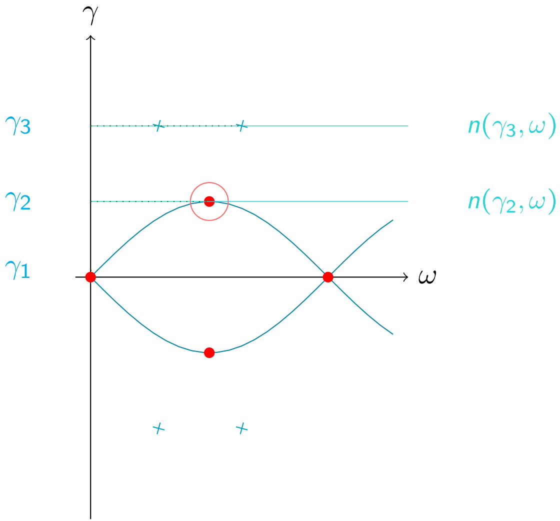
- $\mathcal{P} \in \mathbb{Z}[x, y]$,
- $\bar{x} \in V_{\mathbb{R}}(Res(\mathcal{P}, \frac{\partial \mathcal{P}}{\partial y}, y))$,
- $\mathcal{G} = gcd(\mathcal{P}(\bar{x}, y), \frac{\partial \mathcal{P}}{\partial y}(\bar{x}, y)) \in \mathbb{R}[y]$.

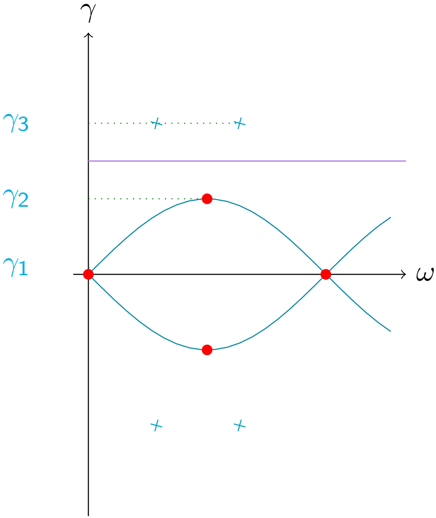
If the x-projection of $\mathcal{P} = 0$ is bounded by $\bar{x}$, then

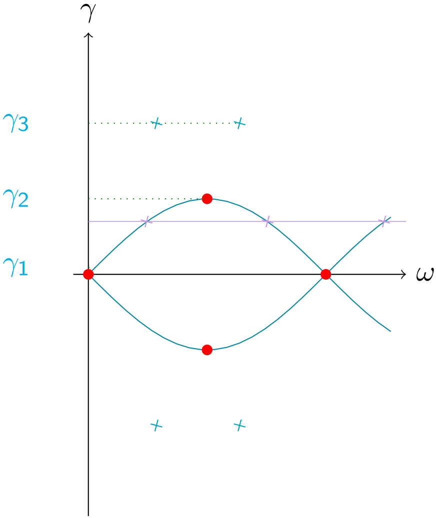
$$\boxed{V_{\mathbb{R}}(\mathcal{P}(\bar{x}, y)) = V_{\mathbb{R}}(\mathcal{G}(\bar{x}, y))}$$


γ
γ_3
γ_2
γ_1
ω

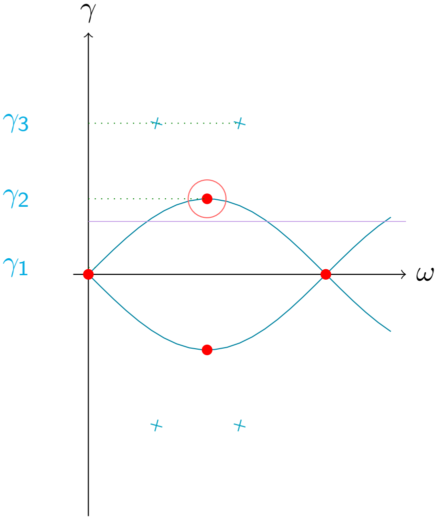
Sturm-Habicht Method

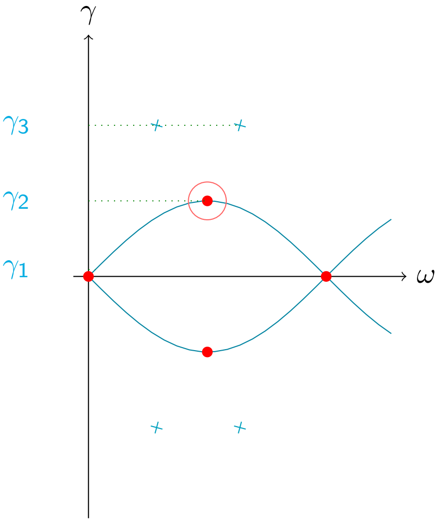



γ
γ3
γ2
γ1
ω



γ
γ_3
γ_2
γ_1
ω


γ
γ3
γ2
γ1
ω



$$||G||_\infty = \gamma_2$$

Symbolic-numeric algorithms

- RUR method
  - $\tilde{\mathcal{O}}_B(\alpha^9 N^4(\alpha + \tau_n))$
- Roots Separation method
  - $\tilde{\mathcal{O}}_B(\alpha^{14} N^5 \tau_n)$
- Sturm-Habicht method
  - i) $\tilde{\mathcal{O}}_B(\alpha^{10} N^4 \tau_n)$
  - ii) $\tilde{\mathcal{O}}_B(\alpha^{10} N^4(\alpha + \tau_n))$

⋆ $G = (\frac{P_{i,j}}{Q_{i,j}})_{i,j} \in \mathcal{R}L_\infty^{u \times v}$, $n(\gamma, \omega) = \texttt{numer}(\det(\Phi_\gamma(s)))$,

⋆ $\deg_\gamma(n) = \mathcal{O}(\alpha)$, $\deg_\omega(n) = \mathcal{O}(\alpha^2 N)$ and $\tau_n = \tilde{\mathcal{O}}(\alpha^2 \tau_G)$.

Proposition

- $\omega_0 \in \mathbb{R}_{+*}$, $\omega_1 \in \mathbb{R}_{+*}$, $\omega_0 \neq \omega_1$, $r := \frac{\omega_1}{\omega_0}$, $0 < \xi \leq 1$
- $R := \frac{(\frac{s}{\omega_0})^2+2\xi(\frac{s}{\omega_0})+1}{(\frac{s}{\omega_1})^2+2\xi(\frac{s}{\omega_1})+1} \in \mathcal{R}L_\infty$
- $\delta = \max_\gamma(V_{\mathbb{R}}((4\xi^4 - 4\xi^2)\gamma^4 + (-8r^2\xi^4 + r^4 + 8r^2\xi^2 - 2r^2 + 1)\gamma^2 + 4r^4\xi^4 - 4r^4\xi^2)$

$$||R||_\infty = \begin{cases} 1 & \text{if} \quad \xi \geq \frac{\sqrt{2}}{2} \\ \delta & \text{if} \quad \xi < \frac{\sqrt{2}}{2} \end{cases}$$

For $\omega_0 := 1$, $\omega_1 := 1.03$, $\xi := 1.08\ 10^{-2}$: `S := Σ`

1. By the proposition: $\|R\|_\infty = 3.155785135$
2. By our proposed methods:

```
Hinf_RUR(S);
```

$\Gamma = 3.155785135$

```
Hinf_Sep(S);
```

[3.155785135, 3.155785135]

```
Hinf_Sres(S);
```

[3.155785135, 3.155785135]

3. By `NormHinf` provided by `DynamicSystems`:

```
NormHinf(S);
```

3.15578801942604

- We consider the matrix G:

$$\begin{bmatrix} \frac{2s-3}{s^2-3s-3} & \frac{s}{-4s^2-3s+3} & \frac{-3s-3}{-3s^2-4s-2} \\ 0 & \frac{3+2s}{-3s^2-s+2} & \frac{2s+1}{3s^2-2} \\ \frac{4}{3s^2+4s-4} & \frac{2s}{-s^2+s+1} & -\frac{3}{4s^2-4s+4} \end{bmatrix}$$

```
>
> S := TransferFunction(G):
  NormHinf(G);
Warning, 'sys' is unstable. Unstable eigenvalues of 'sys': .5687293044, .6666666667,
.8164965809, 1.618033988, 3.791287848, .5000000000-.8660254040*I,
.5000000000+.8660254040*I
Error, (in simpl/max) complex argument to max/min: (1/12)*(3*(68931+(6*I)*2496675387
(1/2))^(1/3)+13671/(68931+(6*I)*2496675387^(1/2))^(1/3)+351)^(1/2)
>
>
> Hinf_RUR(S);
```

$$\Gamma = 2.234750226$$

```
> Hinf_Sep(S);
```

$$[2.234750226, 2.234750226]$$

```
> Hinf_Sres(S);
```

$$[2.234750226, 2.234750226]$$

α	N	Hinf_RUR	Hinf_Sep	Hinf_Sres
2	2	0.2	3	0.2
	3	0.5	7	0.5
	4	2.5	25	2
	5	10	83	6
	6	37	96	10
	7	50	186	47.5
	8	133.5	353	59
	9	236	394	130

$\star \tau_G = 2$

Symbolic-numeric algorithms

- **RUR method**
  - ⋆Separating form
  - ⋆RUR decomposition
  - ⋆Isolating boxes
  - $\tilde{\mathcal{O}}_B(d^6 + d^5\tau)$
- **Roots Separation method**
  - ⋆Separating form
  - ⋆Isolating interval
  - $\tilde{\mathcal{O}}_B(d^9\tau)$
- **Sturm-Habicht method**
  - ⋆ d sign evaluations over i) rational ii) algebraic number
  - i) $\tilde{\mathcal{O}}_B(d^6\tau)$
  - ii) $\tilde{\mathcal{O}}_B(d^7 + d^6\tau)$

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Thank you for your attention