Factoring a Multivariate Polynomial Represented by a Black Box – A Maple to C Implementation

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The black box representation of a polynomial $f \in \mathbb{Z}[x_1, ..., x_n]$

Input: A prime p, $\boldsymbol{\alpha} \in \mathbb{Z}_p^n$
Output: $f(\boldsymbol{\alpha})$ mod p

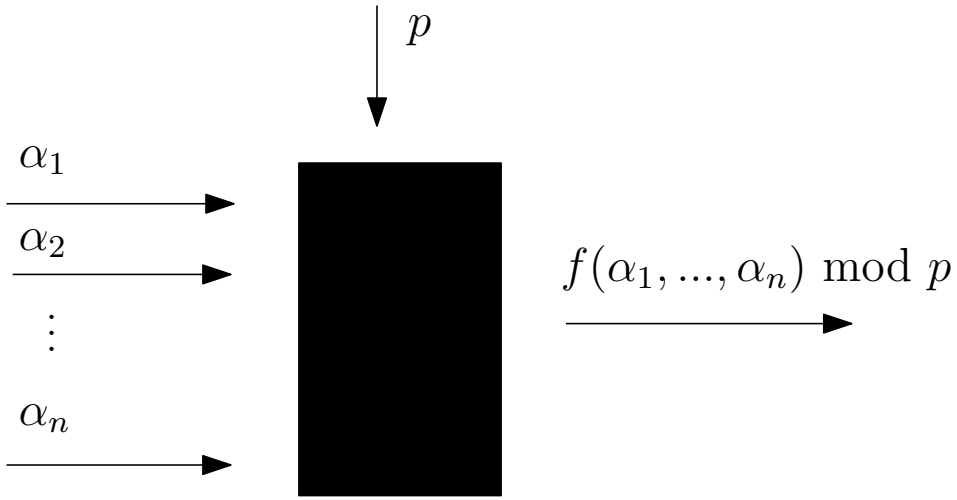
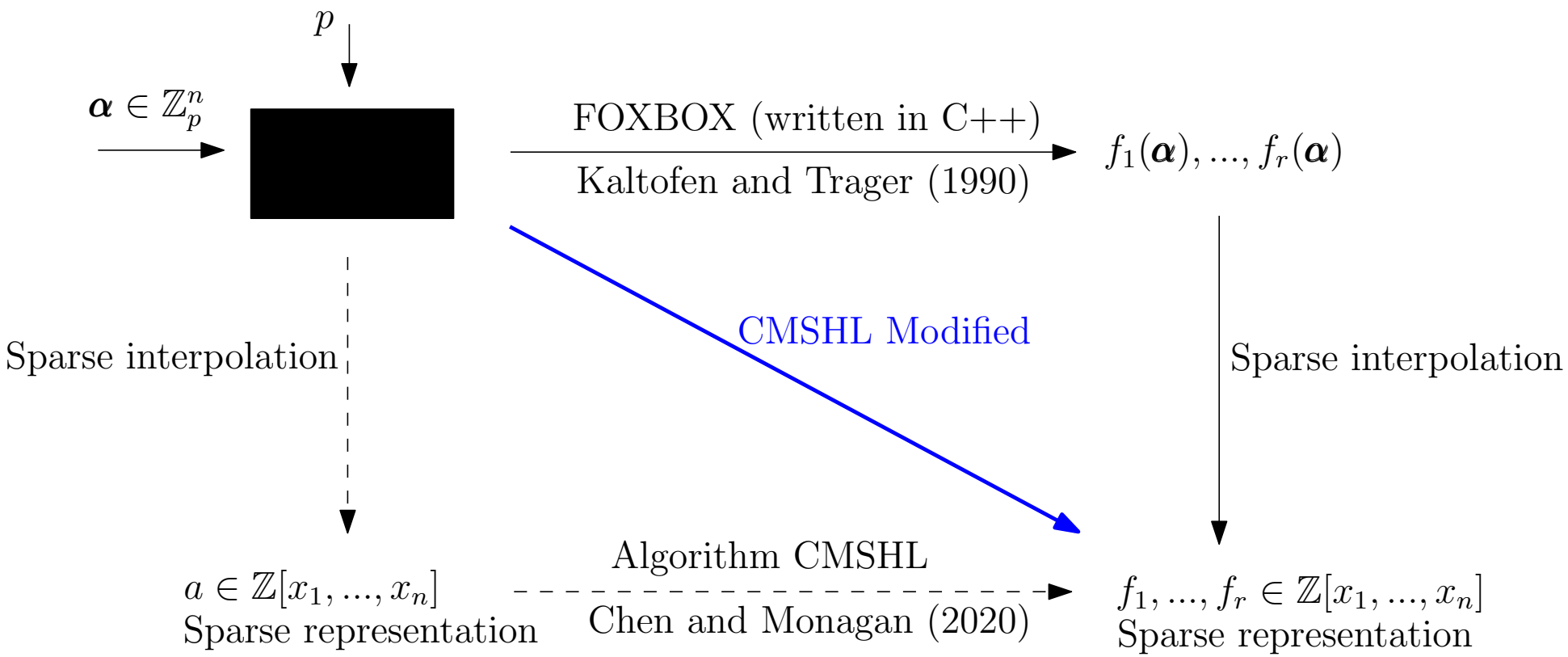


Figure: The black box representation of $f \in \mathbb{Z}[x_1, ..., x_n]$.

E.g. Let A be a matrix with multivariate polynomial entries.
We represent $\det(A)$ as a black box procedure $\mathbf{B}(\boldsymbol{\alpha}, p)$.

Black box representation is space efficient.

Factoring a multivariate polynomial $a \in \mathbb{Z}[x_1, ..., x_n]$



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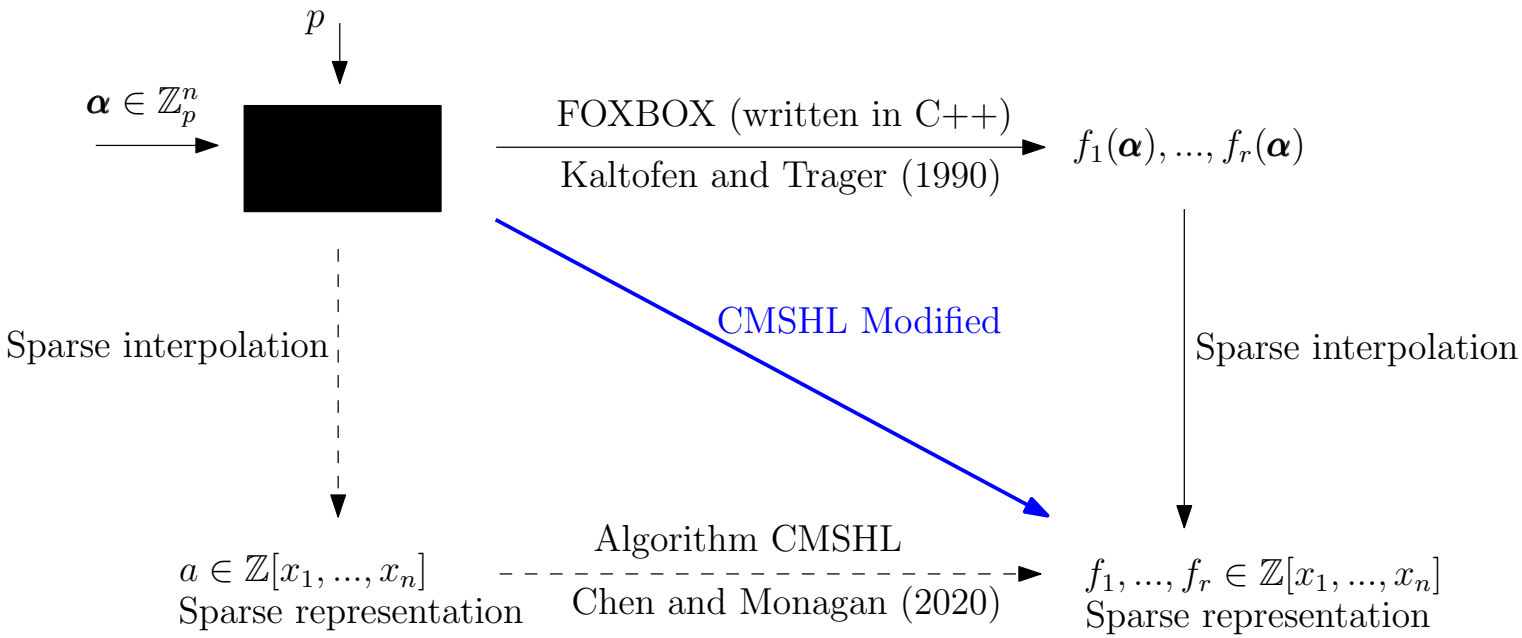

- Diaz and Kaltofen (1998): FOXBOX written in C++
- Lee (2001): PROTOBOX written entirely in Maple
- Our new approach:
  - Maple to C (subroutines in C to speed up the computation)
  - Computes directly from the black box to the factors in sparse representation
  - Extended algorithm CMSHL [Chen and Monagan (2020)] to multifactors

Outline

- Background/Motivation
- Analysis: How many probes to the blackbox?
- Implementation Details
- Code Demo
- Current and Future Work

How many probes to the black box for each approach?

- Method I: Obtain evaluations of the factors, and perform sparse interpolation [Kaltofen and Tragar (1990)].

  $O(nd_1 d_{\max} \# f_{\max})$ probes of $\mathbf{B}$ using Zippel's sparse interpolation, plus $O(nd_{\max} \# f_{\max})$ times the cost of univariate polynomial factorization.

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- Method II: Modified CMSHL [Chen and Monagan (2020)].

  $O(nd_1 d_{\max} s)$ probes of $\mathbf{B}$, to get bi-variate images of a.
  Total cost is:
  $O\left((n-2)\left(s_{\max} d_{\max}\left(\sum_{i=1}^{r} \# f_i + d_1^2 + d_1 d_{\max}\right) + s_{\max} d_1 d_{\max} C(\text{probe } \mathbf{B})\right)\right)$.

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From experiments, we observed that $s \ll \# f_i$, $i = 1..r$, where s is the number of bi-variate images to be interpolated.

Implementation Details

Already implemented Method II in Maple for monic polynomials.

Subroutines for black box

- BB := proc(X::list, alpha::list, p::prime)
  (contains a subroutine to compute determinant in $\mathbb{Z}_p$ in C)
- degBB := proc(X::list, p::prime, j::posint)

Subroutines for CMSHL (Method II)

- Evaluation (done with black box)
  denseinterp := proc(X::list, beta::list, index::integer, p::prime)
- Multifactor BHL (in the middle of integrating it)
- Vandermonde Solve (to be implemented in C)

Current and Future Work

- To integrate more components in C, i.e. BHL (multifactor) and Vandermonde Solve.
- Investigate the efficiency of black box procedure.
- Compare both method I and II.

In the future, may include the non-monic case for completeness.

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Thank you for attending!

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