Using Maple to investigate integer length-preserving directions

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All pictures in this presentation were created using Maple.

Overview

Introduction

In the 2015 Webinar talk *Eigenpairs in Maple* [3], Dr. Robert Lopez discussed how to use Maple to find eigenvalues and eigenvectors (eigenpairs) of a matrix A. An eigenvector of A is a (nonzero) vector whose direction is preserved under multiplication by A.

As an aside, by the end of the talk, Dr. Lopez, posed the question: what about preserving the *magnitude* of the vector, rather than its direction? In other words, what about (nonzero) vectors v such that $\|\mathrm{v}\| = \|A\mathrm{v}\|$, where $\|\mathrm{v}\|$ is the usual Euclidean norm?

Introduction

In search of an example of a matrix with integer coordinates, Dr. Lopez used a "for loop" in Maple, and came up with the following example:

$$A = \begin{pmatrix} 4 & 3 \\ -2 & -3 \end{pmatrix}.$$

Regarded as a map from $\mathbb{R}^2$ to itself, this matrix preserves the norms, but not the directions, of the vectors with integer coordinates $\mathrm{v}_1 = \langle 1, -1 \rangle$ and $\mathrm{v}_2 = \langle 17, -19 \rangle$.

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Indeed, for example,

$$\|v_1\| = \sqrt{17^2 + (-19)^2} = \sqrt{650} = 5\sqrt{26},$$

whereas

$$Av_1 = \begin{pmatrix} 4 & 3 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} 17 \\ -19 \end{pmatrix} = \begin{pmatrix} 11 \\ 23 \end{pmatrix},$$

and

$$\|Av_1\| = \sqrt{11^2 + 23^2} = \sqrt{650} = 5\sqrt{26}.$$

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Another nice fact about this matrix is that its eigenvectors can also be expressed in integer coordinates; they are

$$\mathrm{u}_1 = \langle -1, 2 \rangle \qquad \text{and} \qquad \mathrm{u}_2 = \langle -3, 1 \rangle,$$

with eigenvalues -2 and 3, respectively.

The map generated by A preserves the directions, but not the norms, of the eigenvectors u_1 and u_2.

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Thus, the search for vectors whose norms are preserved by A is in a sense the dual of the search for eigenvectors.

It is not hard to see that a multiple of each of these norm-preserving vectors has the same property. So, just as eigenvectors generate *eigenlines*, these vectors generate what we will call *norm-preserving lines*, or *norm-preserving directions* of the matrix A.

Introduction

Intrigued by Dr. Lopez's example, I wondered if one could find other similar examples of matrices with integer coefficients for which the norm-preserving directions could be generated by vectors having integer coordinates as well.

I ended up finding entire families of such 2×2 matrices, and studied a few examples for the 3×3 case.

General considerations

In general, if a matrix A has eigenvalue ± 1, then the corresponding eigenline is also a norm-preserving line.

Also if A is an orthogonal matrix, then all directions are preserved. In the 2×2 case, these are typically rotations. In this case, all norm-preserving lines will be rotated by the same angle.

However, this is not what happens typically in the general case: norm-preserving lines are rotated by different angles.

The 2×2 case

For a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

a norm-preserving vector $\mathrm{v} = \langle x, y \rangle$ will be a nonzero solution of

$$\|\mathrm{v}\| = \|A\mathrm{v}\|, \tag{1}$$

where

$$\|\mathrm{v}\| = \sqrt{x^2 + y^2}$$

is the usual Euclidean norm.

The 2×2 case

Equation (1) is equivalent to $\|v\|^2 = \|Av\|^2$, or $(v, v) = (Av, Av)$, where (v, w) is the usual Euclidean inner product. The right-hand side becomes

$$\|Av\|^2 = (Av, Av) = (v, A^t Av) = (v, Bv),$$

where A^t is the transpose of A, and $B = A^t A$. Thus, equation (1) is equivalent to

$$(v, (B - I)v) = 0, \tag{2}$$

Where I is the identity matrix.

Further, we have

$$B = A^t A = \begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix} = \begin{pmatrix} m & p \\ p & n \end{pmatrix},$$

where

$$m = a^2 + c^2, \quad (3)$$
$$n = b^2 + d^2, \quad (4)$$
$$p = ab + cd, \quad (5)$$

The 2×2 case

so that if we denote $\mathrm{v} = \langle x, y \rangle$, then the quadratic form at the left-hand side of (2) is

$$\Phi(x, y) = (m-1)x^2 + 2pxy + (n-1)y^2; \tag{6}$$

with this notation, (1) or, equivalently, (2), is in turn equivalent to $\Phi(x, y) = 0$.

The 2×2 case

This is our first result:

Theorem

Norm-preserving lines exist if and only if

$$a^2 + b^2 + c^2 + d^2 \geq 1 + \det(A)^2. \tag{7}$$

The 2×2 case

Without getting into details, let me just comment that the proof is a consequence of the following geometric argument.

The eigenvalues λ_1 and λ_2 of the symmetric matrix $B = A^t A$ are (real and) nonnegative; assume $0 \leq \lambda_1 \leq \lambda_2$. By the extreme properties of eigenvalues (see, for example, [1] or [4]), we have

$$\begin{aligned} \lambda_1 &= \min_{\|\mathrm{v}\|=1} \|A\mathrm{v}\|^2 = \min_{\|\mathrm{v}\|=1} (\mathrm{v}, B\mathrm{v}) \\ &\leq \max_{\|\mathrm{v}\|=1} \|A\mathrm{v}\|^2 = \max_{\|\mathrm{v}\|=1} (\mathrm{v}, B\mathrm{v}) = \lambda_2. \end{aligned}$$

The 2×2 case

Therefore, there will exist norm-preserving lines v such that $\|v\| = \|Av\|$ if and only if

$$\lambda_1 \leq 1 \leq \lambda_2.$$

When the eigenvalues are strictly positive, this condition guarantees the intersection of the ellipse $(v, Bv) = 1$, for which the half-axes are

$$\frac{1}{\sqrt{\lambda_1}} \quad \text{and} \quad \frac{1}{\sqrt{\lambda_2}},$$

with the unit circle $\|v\| = 1$.

The 2×2 case

The following picture shows the ellipse and the unit circle for the case of the matrix A in R. Lopez's example:

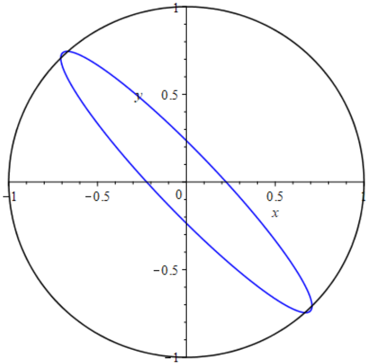


Figure: Illustration of R. Lopez's example.

Families of integer matrices

Let us assume that the entries a, b, c, d of our 2×2 matrix A are integers. Finding norm-preserving lines generated by vectors $\mathrm{v} = \langle x, y \rangle$ with integer coordinates leads us to solving the Diophantine equation

$$\Phi(x, y) = (m-1)x^2 + 2pxy + (n-1)y^2 = 0,$$

where m, n, p are expressed in terms of a, b, c, as shown before.

Families of integer matrices

Working with this expression, we were able to find the following four families of matrices with integer norm-preserving lines:

$$\begin{pmatrix} a & a \pm 1 \\ c & c \pm 1 \end{pmatrix},$$

where a and c are arbitrary integers.
Their transposes, by the way, are also good:

$$\begin{pmatrix} a & c \\ a \pm 1 & c \pm 1 \end{pmatrix}.$$

Families of integer matrices

The example provided by R. Lopez corresponds to the family

$$A = \begin{pmatrix} a & a-1 \\ c & c-1 \end{pmatrix}, \tag{8}$$

when we choose $a = 4$ and $c = -2$.
For the general case of matrices in this particular family, one can show that the norm-preserving lines are determined by the integer vectors

$$\mathsf{v}_1 = \langle 1, -1 \rangle$$

and

$$\mathsf{v}_2 = \langle (a-1)^2 + (c-1)^2 - 1, 1 - a^2 - c^2 \rangle.$$

Families of integer matrices

For example, for $a = 2$ and $c = -3$, we get the matrix

$$A = \begin{pmatrix} 2 & 1 \\ -3 & -4 \end{pmatrix},$$

with norm-preserving lines determined by $\mathrm{v}_1 = \langle 1, -1 \rangle$ and $\mathrm{v}_2 = \langle 4, -3 \rangle$.

Families of integer matrices

Here is the corresponding picture, showing the direction lines which, as expected, pass through the intersections of the ellipse $\|Av\| = 1$ with the unit circle.

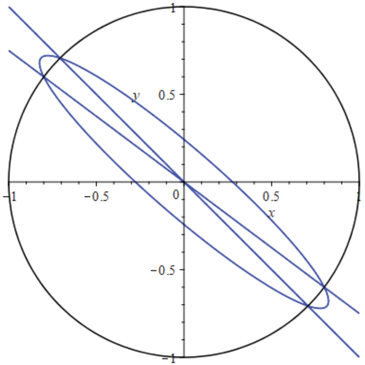


Figure: Norm-preserving directions shown.

The 3×3 case.

The 3×3 case is considerably more complicated, as well as more interesting.
If A is a 3×3 real-valued matrix, also regarded as a linear map from $\mathbb{R}^3$ to itself, then in general $\|A\mathrm{v}\|^2 = 1$ is an ellipsoid, in terms of the coordinates of $\mathrm{v} = \langle x, y, z \rangle$. As in the 2×2 case, we have

$$\|A\mathrm{v}\|^2 = (A\mathrm{v}, A\mathrm{v}) = (\mathrm{v}, B\mathrm{v}),$$

where $B = A^t A$ is a symmetric matrix with nonnegative eigenvalues. The equation for norm-preserving vectors, $\|A\mathrm{v}\| = \|\mathrm{v}\|$, or $(\mathrm{v}, B\mathrm{v}) = (\mathrm{v}, \mathrm{v})$, is equivalent to the cone

$$(\mathrm{v}, (B - I)\mathrm{v}) = 0, \tag{9}$$

where I is the identity matrix.

The 3×3 case.

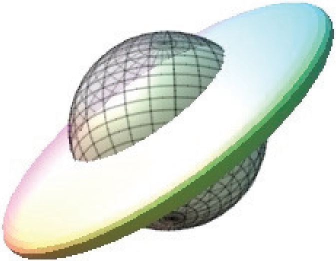
As in the 2×2 case, if we denote the eigenvalues of B by $0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3$, then there is a solution of (9) if and only if

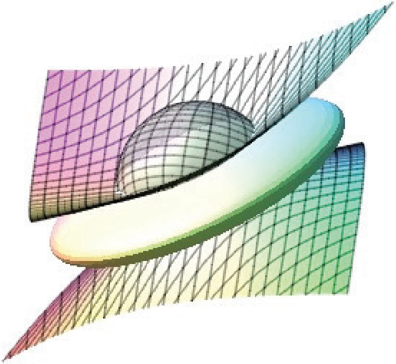
$$\lambda_1 \leq 1 \leq \lambda_3, \tag{10}$$

which guarantees a nonempty intersection of the ellipsoid (or degenerate ellipsoid) $\|B\mathrm{v}\|^2 = 1$ with the unit sphere $\|\mathrm{v}\|^2 = 1$.
If condition (10) is satisfied, then the cone determined by (9) will pass through this intersection of the ellipsoid and the unit sphere.

The 3×3 case.

Here is an illustration, corresponding to the matrix A mentioned below in Example 1: the figure on the left shows the ellipsoid $\|B\mathrm{v}\|^2 = 1$ and the unit sphere; and on the right, we add the cone $(\mathbf{v}, (\mathrm{B}\text{-}\mathbf{I})\mathbf{v}) = 0$; compare with the corresponding pictures for the 2×2 case.





The 3×3 case.

Let us explore a few examples which show what one could have when looking for integer norm-preserving directions for specific matrices with integer or rational coefficients.

Example 1. Let us consider the symmetric matrix

$$A = \begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 1 & \frac{1}{2} & 1 \\ \frac{1}{2} & 1 & 1 \end{pmatrix} \tag{11}$$

The 3×3 case.

The form $\|A\mathrm{v}\|^2 = (A\mathrm{v}, A\mathrm{v}) = (\mathrm{v}, B\mathrm{v})$ will in this case have matrix

$$B = A^t A = A^2 = \begin{pmatrix} \frac{9}{4} & 2 & 2 \\ 2 & \frac{9}{4} & 2 \\ 2 & 2 & \frac{9}{4} \end{pmatrix}$$

so that equation (9) will be

$$\frac{5}{4}x^2 + 4xy + 4xz + \frac{5}{4}y^2 + 4yz + \frac{5}{4}z^2 = 0. \qquad (12)$$

This equation has infinitely many real-valued solutions, which constitute the solution cone shown in the previous picture.

The 3×3 case.

However, the equation has no nonzero integer solutions.
To prove this, we solve the quadratic equation for z (12) and get

$$z = -\frac{8}{5}(x+y) \pm \frac{\sqrt{39x^2 + 48xy + 39y^2}}{5}.$$

Therefore, there will be integer solution lines if and only if the discriminant $39x^2 + 48xy + 39y^2$ is a perfect square. Our second result, which involves working with a quadratic Diophantine equation, is:

Theorem

The Diophantine equation

$$39x^2 + 48xy + 39y^2 = u^2 \tag{13}$$

has no nontrivial solutions, that is, no nonzero integer solutions.

The 3×3 case.

Example 2: a dense set of integer solution lines.
Consider the symmetric matrix

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}. \tag{14}$$

Here

$$B = A^t A = A^2 = \begin{pmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{pmatrix}$$

The 3×3 case.

so that equation (9) will be

$$8x^2 + 16xy + 16xz + 8y^2 + 16yz + 8z^2 = 0,$$

or (after dividing by 8)

$$(x + y + z)^2 = 0. \tag{15}$$

In our case, the cone degenerates into the plane $x + y + z = 0$. We can pick an integer basis, say $\mathrm{v}_1 = \langle 1, 0, -1 \rangle$ and $\mathrm{v}_2 = \langle 0, 1, -1 \rangle$, and obtain every integer norm-preserving line as generated by $\alpha \mathrm{v}_1 + \beta \mathrm{v}_2$, with integer coefficients α, β such that $\alpha^2 + \beta^2 > 0$. This constitutes a dense set of integer solution lines, among all possible solutions in the plane $x + y + z = 0$.

The 3×3 case.

The ellipsoid $\|Av\|^2 = 1$ lies inside the unit sphere, and is tangent to it along the intersection of the sphere with the solution plane; here is the corresponding picture:

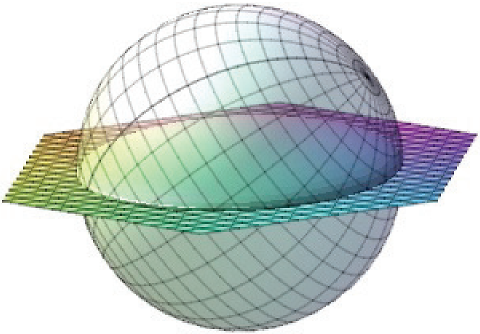


Figure: Solution cone is degenerate.

Example 3: infinitely many integer solution lines.

Finally, let us consider an example when there are still infinitely many integer solution lines, yet we cannot guarantee they are dense in the cone of all solution lines. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{16}$$

One eigenvalue of A is -1, with eigenvector $=\langle -1, 1, 0\rangle$, which therefore provides one integer solution line.

The 3×3 case.

Are there any other such lines? The matrix $B = A^t A$ is

$$B = \begin{pmatrix} 6 & 5 & 6 \\ 5 & 6 & 8 \\ 6 & 8 & 11 \end{pmatrix}$$

and consequently equation (9) becomes

$$5x^2 + 10xy + 12xz + 5y^2 + 16yz + 10z^2 = 0. \tag{17}$$

The 3×3 case.

An analysis of this quadratic Diophantine equation leads to the following result:

Theorem

All integer solutions of (17), besides the eigenline generated by $\langle -1, 1, 0\rangle$, are given by

$$\begin{cases} x = \frac{1}{10}(r^2 - 10v^2 \pm 4vr) \\ y = -\frac{1}{10}(14v^2 + r^2) \\ z = 2v^2, \end{cases}$$

where v and r are arbitrary integers.

The 3×3 case.

For example, choosing $(v, r) = (1, 4)$ we get the vectors

$$\left\langle \frac{11}{5}, -3, 2 \right\rangle \sim \langle 11, -15, 10 \rangle \qquad \text{and} \qquad \langle 1, 3, -2 \rangle;$$

and choosing $(v, r) = (1, 1)$ we obtain

$$\left\langle -\frac{1}{2}, -\frac{3}{2}, 2 \right\rangle \sim \langle 1, 3, -4 \rangle \qquad \text{and} \qquad \left\langle -\frac{13}{10}, -\frac{3}{2}, 2 \right\rangle \sim \langle 13, 15, -20 \rangle.$$

Each of them generates a norm-preserving line for the given matrix.

References

[1] Gel'fand, I. M. (1989). *Lectures on Linear Algebra*. New York: Dover Publications.

[2] Katok A., Hasselblatt B. (1995). *Introduction to the Modern Theory of Dynamical Systems*. New York: Cambridge University Press.

[3] Lopez, R. (2015). *Eigenpairs in Maple*. https://www.maplesoft.com/webinars/recorded/featured.aspx?id=1181.

[4] Shilov, G. E. (1977). *Linear Algebra*. New York: Dover Publications.

[5] Weisstein, E. W. (2019). Diophantine equation–2nd powers. *MathWorld*–A Wolfram Web Resource. mathworld.wolfram.com/DiophantineEquation2ndPowers.html.

Summary

Starting from one example suggested by Dr. Robert Lopez in a Maple webinar, we study directions along which the norms of vectors are preserved under a matrix, regarded as a linear map.

In particular, we find families of 2×2 matrices for which these directions are determined by integer vectors.

We also consider a few examples for 3×3 matrices, to get an idea of what to expect for the 3×3 case.

Maple was an integral part of this research, for calculations, exploration, and constructing pictures both for illustrations and to gain insight into the geometry of the topic.

The End