A Maple Solution to the Problem 6 of the IMO 1988

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(Extended Abstract)

The 29th International Mathematical Olympiad was held on July 9-21, 1988 at Canberra, Australia. There were 268 contestants from 49 countries participated the Olympiad, and 17 people of them got the golden prize. The Problem 6 of IMO 1988 was called as “The Legend Problem 6 of IMO” since only 11 among 268 participants answered it correctly (that means that they obtained 7 points, the highest score, for this question), which is significantly lower than the correct ratios of other problems of this Olympiad. The Problem 6 reads as follows: let a and b be positive integers such that $ab+1$ divides a^2+b^2, show that $\frac{a^2+b^2}{ab+1}$ is the square of an integer. In this paper, we will show an approach to solve this problem with using Maple software as a computation tool.

The maple computation is starting from the following short program:

```
>
  Sab := [ ];
  for a to 1000 do
        for b to 1000 do
            if (a^2 + b^2) mod (a b + 1) = 0 then
                Sab := [op(Sab), [a, b,
                (a^2 + b^2)/(a b + 1)]]
            end if
        end do
     end do;
  Sab
```

It yields the following result in 1 second:

[[1, 1, 1], [2, 8, 4], [3, 27, 9], [4, 64, 16], [5, 125, 25], [6, 216, 36], [7, 343, 49], [8, 2, 4], [8, 30, 4], [8, 512, 64], [9, 729, 81], [10, 1000, 100], [27, 3, 9], [27, 240, 9], [30, 8, 4], [30, 112, 4], [64, 4, 16], [112, 30, 4], [112, 418, 4], [125, 5, 25], [216, 6, 36], [240, 27, 9], [343, 7, 49], [418, 112, 4], [512, 8, 64], [729, 9, 81], [1000, 10, 100]]

The list Sab contains 27 integer triples (a,b,k) that satisfies $a^2+b^2=k(ab+1)$. It is apparent that if (a,b,k) is a solution of this Diophantine equation, then (b,a,k) is also a solution. Try the above program for a from 1 to 10000 and for b from a to 10000, the computation outputs the following 31 triples in about 93 seconds:

[[1, 1, 1], [2, 8, 4], [3, 27, 9], [4, 64, 16], [5, 125, 25], [6, 216, 36], [7, 343, 49], [8, 30, 4], [8, 512, 64], [9, 729, 81], [10, 1000, 100], [11, 1331, 121], [12, 1728, 144], [13, 2197, 169], [14, 2744, 196], [15, 3375, 225], [16, 4096, 256], [17, 4913, 289], [18, 5832, 324], [19, 6859, 361], [20, 8000, 400], [21, 9261, 441], [27, 240, 9], [30, 112, 4], [64, 1020, 16], [112, 418, 4], [125, 3120, 25], [216, 7770, 36], [240, 2133, 9], [418, 1560, 4], [1560, 5822, 4]]

Do further experiment for a from 1 to 100000 and for b from a to 100000. Then, the computation took 16990 seconds (on a desktop computer with Intel ® 2 Duo CPU T7500, 2.0GB RAM, Maple 15). The results contains 65 solutions (a,b,k) with $a \le b$. The result can be rearranged as follows:

[1, 1, 1],

[2, 8, 4], [8, 30, 4], [30, 112, 4], [112, 418, 4], [418, 1560, 4], [1560, 5822, 4], [5822, 21728, 4], [21728, 81090, 4]]

[3, 27, 9], [27, 240, 9], [240, 2133, 9], [2133, 18957, 9],

[4, 64, 16], [64, 1020, 16], [1020, 16256, 16],

[5, 125, 25], [125, 3120, 25], [3120, 77875, 25],

[6, 216, 36], [216, 7770, 36],

[7, 343, 49], [343, 16800, 49],

[8, 512, 64], [512, 32760, 64],

[9, 729, 81], [729, 59040, 81],

[10, 1000, 100], [1000, 99990, 100],

[11, 1331, 121], [12, 1728, 144], [13, 2197, 169], [14, 2744, 196], [15, 3375, 225],

[16, 4096, 256], [17, 4913, 289], [18, 5832, 324], [19, 6859, 361], [20, 8000, 400],

[21, 9261, 441], [22, 10648, 484], [23, 12167, 529], [24, 13824, 576],

[25, 15625, 625], [26, 17576, 676], [27, 19683, 729], [28, 21952, 784],

[29, 24389, 841], [30, 27000, 900], [31, 29791, 961], [32, 32768, 1024],

[33, 35937, 1089], [34, 39304, 1156], [35, 42875, 1225], [36, 46656, 1296],

[37, 50653, 1369], [38, 54872, 1444], [39, 59319, 1521], [40, 64000, 1600],

[41, 68921, 1681], [42, 74088, 1764], [43, 79507, 1849], [44, 85184, 1936],

[45, 91125, 2025], [46, 97336, 2116],

From the above data, it is easily observed that

(1) For any positive integer p,

$$a = p, b = p^3, k = p$$

forms a solution of the Diophantine equation $a^2 + b^2 = k(ab+1)$. Indeed, this property can be checked very easily.

(2) For any positive integer k, if (a,b,k) is a solution of the Diophantine equation, then it seems that for some integer $b' > b$, (b,b',k) also a solution of the equation, which means, it is possible that for any integer k with $k = p^2$ where p is an integer and $p > 1$, there is a sequence of integers

$$s_0(k) = p < s_1(k) = p^3 < s_2(k) < \cdots < s_n(k) < \cdots$$

So that all triples

$$(s_0, s_1, k), (s_1, s_2, k), \cdots, (s_n, s_{n+1}, k), \cdots$$

are solutions of $a^2 + b^2 = k(ab+1)$. If this is actually true, then we have

$$s_{n-1}^2 + s_n^2 = k(s_{n-1}s_n + 1), \quad s_n^2 + s_{n+1}^2 = k(s_n s_{n+1} + 1),$$

and therefore,

$$s_{n+1}^2 - s_{n-1}^2 = k(s_{n+1} - s_{n-1})s_n,$$

thus, $s_{n+1} = -s_{n-1} + ks_n$ for all integers $n = 1,2,\cdots$. Formally, we can prove a proposition as follows: if (a,b,k) is a solution of the Diophantine equation $a^2 + b^2 = k(ab+1)$, then $(ka-b,a,k),(b,-a+kb,k)$ are also solutions of this equation, here we don't need the condition $k = p^2$ for some integer $p > 1$ in the proof. For the original Problem 6 of IMO 1988, where it is required that $a,b > 0$, so the minimal element (a,b,k) that satisfies $a^2 + b^2 = k(ab+1)$ must also satisfies condition $b - ak = 0$, thus,

$$k = (a^2 + b^2)/(ab+1) = (a^2 + k^2a^2)/(ka^2 + 1),$$

which implies that $k + k^2a^2 = a^2 + k^2a^2$, that is, $k = a^2$.

The above Maple experiment shows that sufficient numeric solutions of a very difficult Diophantine equation, like the Problem 6 of IMO 1988, could lead to a correct answer to the question. In the rest of this paper, we show some maple computation for the general solution of $s_n(k), n = 1,2,\cdots$. As we know, for $k = p^2$, the initial values are $s_1 = p,\ \ s_2 = p^3$, applying the recurrence relation $s_{n+1} = -s_{n-1} + p^2 s_n$, we have the following terms:

$$\begin{aligned}
s_3 &= p^5 - p,\\
s_4 &= p^7 - 2p^3,\\
s_5 &= p^9 - 3p^5 + p,\\
s_6 &= p^{11} - 4p^7 + 3p^3,\\
s_7 &= p^{13} - 5p^9 + 6p^5 - p,\\
s_8 &= p^{15} - 6p^{11} + 10p^7 - 4p^3,\\
s_9 &= p^{17} - 7p^{13} + 15p^9 - 10p^5 + p,\\
s_{10} &= p^{19} - 8p^{15} + 21p^{11} - 20p^7 + 5p^3,\\
s_{11} &= p^{21} - 9p^{17} + 28p^{13} - 25p^9 + 15p^5 - p,\\
s_{12} &= p^{23} - 10p^{19} + 36p^{15} - 56p^{11} + 35p^7 - 6p^3,\\
s_{13} &= p^{25} - 11p^{21} + 45p^{17} - 84p^{13} + 70p^9 - 21p^5 + p,\\
s_{14} &= p^{27} - 12p^{23} + 55p^{19} - 120p^{15} + 126p^{11} - 56p^7 + 7p^3,\\
s_{15} &= p^{29} - 13p^{25} + 66p^{21} - 165p^{17} + 210p^{13} - 120p^9 + 28p^5 - p,
\end{aligned}$$

and so on. In fact, from the recurrence relation $s_{n+1} = -s_{n-1} + p^2 s_n$, we may have

$$s_n = \sum_{0 \le i \le (2n-1)/4} (-1)^i \binom{n-1-i}{i} p^{(2n-1)-4i}.$$

We will discuss how to recover this formula from several particular instances of s_n.

To do this, we compare the coefficients of $s_1, s_2, s_3, s_4, \ldots$ with the Yang Hui's (Pascal's) triangle:

1									
1	1								
1	2	1							
1	3	3	1						
1	4	6	4	1					
1	5	10	10	5	1				
1	6	15	20	15	6	1			
1	7	21	35	35	21	7	1		
1	8	28	56	70	56	28	8	1	
1	9	36	84	126	126	84	36	9	1

We observe that (the absolute value of) every coefficient of s_n seems an element of the triangle, and for each fixed s_n, the abstract of its coefficients are contained in certain straight line. For example, the coefficients 1, -5, 6, -1 of s_7 appear in green in the Yang Hui triangle, and the red numbers 1, 9, 21, 20, 5 are corresponding to the coefficients of s_{10}. This inspired to find a formula like

$$s_n = \sum_{0 \le i \le (2n-1)/4} (-1)^i \binom{n-1-i}{i} p^{(2n-1)-4i}.$$

We will show how to find such pattern from the Yang Hui's (Pascal's) Triangle with a Maple program for given polynomial series in the final part.

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