Introduction to Vibrations

Free Response Part 2: Spring-Mass Systems with Damping

The equations for the spring-mass model, developed in the previous module (Free Response Part 1), predict that the mass will continue oscillating indefinitely. Through experience we know that this is not the case for most situations. In this tutorial we will introduce the concept of viscous damping to account for decaying motion and study the different types of responses that can result. This module is a continuation of Free Response Part 1: Spring-mass systems.

▼ Spring-Mass Model with Viscous Damping

To modify the equations of motion to account for decaying motion, an additional term is added that is proportional to the velocity $\dot{x}$. This term is in the form $c \cdot \dot{x}(t)$ where c is a constant and is called the damping coefficient (or damping constant). This damping corresponds to the type of resistance to motion and energy dissipation that is encountered when a piston with perforations is moved through a cylinder filled with a viscous fluid, for example oil. Air drag at low velocities, internal forces in structures like shafts and springs, etc. can be approximated using this form where the opposing force is directly proportional to the velocity.

Returning to the horizontal spring-mass system and adding a damper to it, as shown in Fig. 1, we get the following equation by summing the forces in the x-direction.

$$m \cdot \ddot{x}(t) = -c \cdot \dot{x}(t) - k \cdot x(t)$$

... Eq. (1)

or

$$m \cdot \ddot{x}(t) + c \cdot \dot{x}(t) + k \cdot x(t) = 0$$
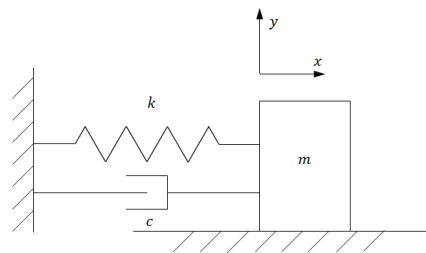
... Eq. (2)

Fig. 1: Single-degree-of-freedom with damping

This equation can be solved using the same method used to solve the differential equation for the spring-mass system in Part 1.

Assuming that the solution has the form $a \cdot e^{\lambda \cdot t}$, and substituting it into Eq. (2) we get

$$\left(m \cdot \lambda^2 + c \cdot \lambda + k \right) \cdot a \cdot e^{\lambda \cdot t} = 0$$

... Eq. (3)

Since $a \cdot e^{\lambda \cdot t}$ is non-zero, we get

$$m \cdot \lambda^2 + c \cdot \lambda + k = 0$$

... Eq. (4)

The solution to this equation is

$$\lambda_{1,2} = - \frac{c}{2 \cdot m} \pm \frac{\sqrt{c^2 - 4 \cdot k \cdot m}}{2 \cdot m}$$

... Eq. (5)

Examining this equation leads us to conclude that, depending on the value of $c^2 - 4 \cdot k \cdot m$, the roots $\lambda_{1,2}$ will either be real or complex. The damping coefficient for which the expression under the square root is equal to zero is defined as the critical damping coefficient c_{cr}, and is given by

$$c_{cr} = 2\sqrt{k \cdot m} = 2 \cdot m \cdot \omega_n$$

... Eq. (6)

Also, the dimensionless number ζ, called the damping ratio, is defined such that

$$\zeta = \frac{c}{c_{cr}} = \frac{c}{2\sqrt{k \cdot m}} = \frac{c}{2 \cdot m \cdot \omega_n}$$

... Eq. (7)

Eq. (5) can be rewritten as

$$\lambda_{1,2} = -\zeta \cdot \omega_n \pm \omega_n \cdot \sqrt{\zeta^2 - 1}$$

... Eq. (8)

The damping ratio allows us to determine whether the roots $\lambda_{1,2}$ are real or complex, which determines the type of response the system displays.

▼ Underdamped motion (0 < ζ < 1)

In this case, the roots are a pair of complex conjugates and can be written as

$$\lambda_1 = -\zeta \cdot \omega_n - \omega_n \cdot \sqrt{1 - \zeta^2} \cdot j$$

... Eq. (9)

and

$$\lambda_2 = -\zeta \cdot \omega_n + \omega_n \cdot \sqrt{1 - \zeta^2} \cdot j$$

... Eq. (10)

Using the same steps used to arrive at the solution for the spring-mass system, the solution for the spring-mass-damper system can be written as

$$x(t) = e^{-\zeta \cdot \omega_n \cdot t} \cdot \left(a_1 \cdot e^{j\sqrt{1-\zeta^2} \cdot \omega_n \cdot t} + a_2 \cdot e^{-j\sqrt{1-\zeta^2} \cdot \omega_n \cdot t} \right)$$

... Eq. (11)

which, using the Euler identity, can be written as

$$x(t) = A \cdot e^{-\zeta \cdot \omega_n \cdot t} \cdot \sin\left(\omega_d \cdot t + \phi \right)$$

... Eq. (12)

where A and ϕ are constants that can be determined using the initial conditions. Additionally, the term ω_d, called the damped natural frequency, is defined as

$$\omega_d = \omega_n \cdot \sqrt{1 - \zeta^2}$$

Using the initial conditions,

$$x_0 = x(0) = A \cdot \sin(0 + \phi)$$

and

$$v_0 = v(0) = -\zeta \cdot \omega_n \cdot A \cdot \sin(0 + \phi) + \omega_d \cdot A \cdot \cos(0 + \phi)$$

we can obtain the following equations for the two unknowns A and ϕ

$$A = \sqrt{\frac{\omega_d^2 \cdot x_0^2 + \left(v_0 + \zeta \cdot \omega_n \cdot x_0\right)^2}{\omega_d^2}}$$

and

$$\phi = \tan^{-1}\left(\frac{\omega_d \cdot x_0}{v_0 + \zeta \cdot \omega_n \cdot x_0}\right)$$

The form of Eq. (12) shows that the position has a sinusoidal vibration that decays over time.

The following plot shows the system response for a mass-spring-damper system with $\zeta = 0.2$.
$(m = 0.1\ kg,\ c = 0.5\ N \cdot s \cdot m^{-1},\ k = 40\ N \cdot m^{-1},\ x_0 = 0.1\ m,\ v_0 = 10\ ms^{-1})$

Response for damping ratio=0.2
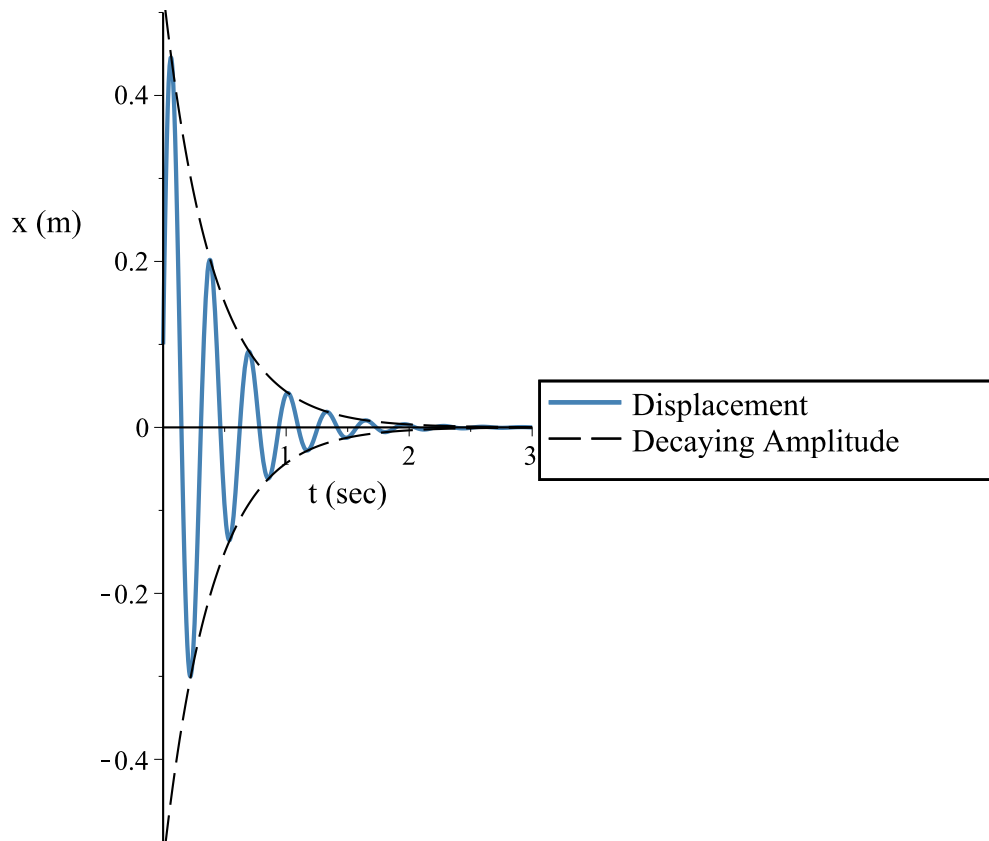
From this plot it can be seen that the amplitude of the vibration decays over time.

▼ Overdamped motion ($\zeta > 1$)

In this case, the roots are a pair of real numbers and can be written as

$$\lambda_1 = -\zeta \cdot \omega_n - \omega_n \cdot \sqrt{\zeta^2 - 1}$$

... Eq. (18 a)

and

$$\lambda_2 = -\zeta \cdot \omega_n + \omega_n \cdot \sqrt{\zeta^2 - 1}$$

... Eq. (18 b)

Using the same steps used to arrive at the solution for the spring-mass system, the solution for the spring-mass-damper system can be written as

$$x(t) = e^{-\zeta \cdot \omega_n \cdot t} \cdot \left(a_1 \cdot e^{-\sqrt{\zeta^2 - 1} \cdot \omega_n \cdot t} + a_2 \cdot e^{\sqrt{\zeta^2 - 1} \cdot \omega_n \cdot t} \right)$$

... Eq. (19)

Since there are no complex numbers, this represents a nonoscillatory response. In this case the constants a_1 and a_2 are given by

$$a_1 = \frac{-v_0 + \left(-\zeta + \sqrt{\zeta^2 - 1} \right) \cdot \omega_n \cdot x_0}{2 \cdot \omega_n \cdot \sqrt{\zeta^2 - 1}}$$

... Eq. (20)

and

$$a_2 = \frac{v_0 + \left(\zeta + \sqrt{\zeta^2 - 1} \right) \cdot \omega_n \cdot x_0}{2 \cdot \omega_n \cdot \sqrt{\zeta^2 - 1}}$$

... Eq. (21)

The following plot shows the system response for a mass-spring-damper system with $\zeta = 2$. The initial conditions and system parameters for this curve are the same as the ones used for the underdamped response shown in the previous section except for the damping coefficient which is 16 times greater.

$(m = 0.1 \ kg, \ c = 8 \ N \cdot s \cdot m^{-1}, \ k = 40 \ N \cdot m^{-1}, \ x_0 = 0.1 \ m, \ v_0 = 10 \ ms^{-1})$

Response for damping ratio=2
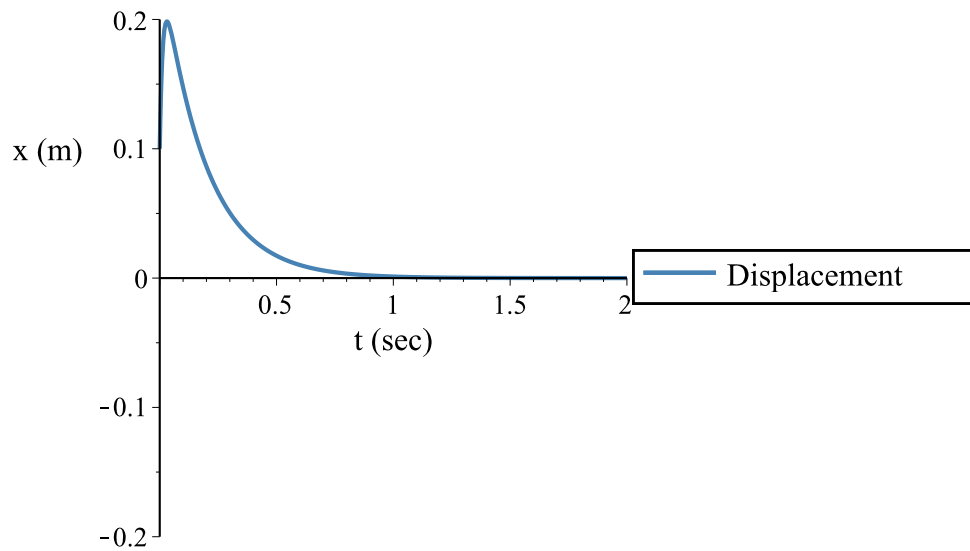
From this plot we can see that the response decays more quickly than the underdamped response shown in the previous section. The larger damping leads to quicker energy dissipation and does not allow the system to vibrate.

▼ Critically Damped motion (ζ=1)

In this case, the damping ratio is equal to 1 and the roots of Eq. (4) are

$$\lambda_1 = \lambda_2 = -\omega_n$$

... Eq. (22)

The solution for the system is

... Eq. (23)

where the constants a_1 and a_2 are given by

$$a_1 = x_0$$

and

$$a_2 = v_0 + \omega_n \cdot x_0$$

The following plot shows the system response for a mass-spring-damper system with $\zeta = 1$. The initial conditions and system parameters for this curve are the same as the ones used for the underdamped and overdamped responses shown in the previous sections except for the damping coefficient.

$$(m = 0.1 \ kg, \ c = 4 \ N \cdot s \cdot m^{-1}, \ k = 40 \ N \cdot m^{-1}, \ x_0 = 0.1 \ m, \ v_0 = 10 \ ms^{-1})$$

Response for damping ratio=1
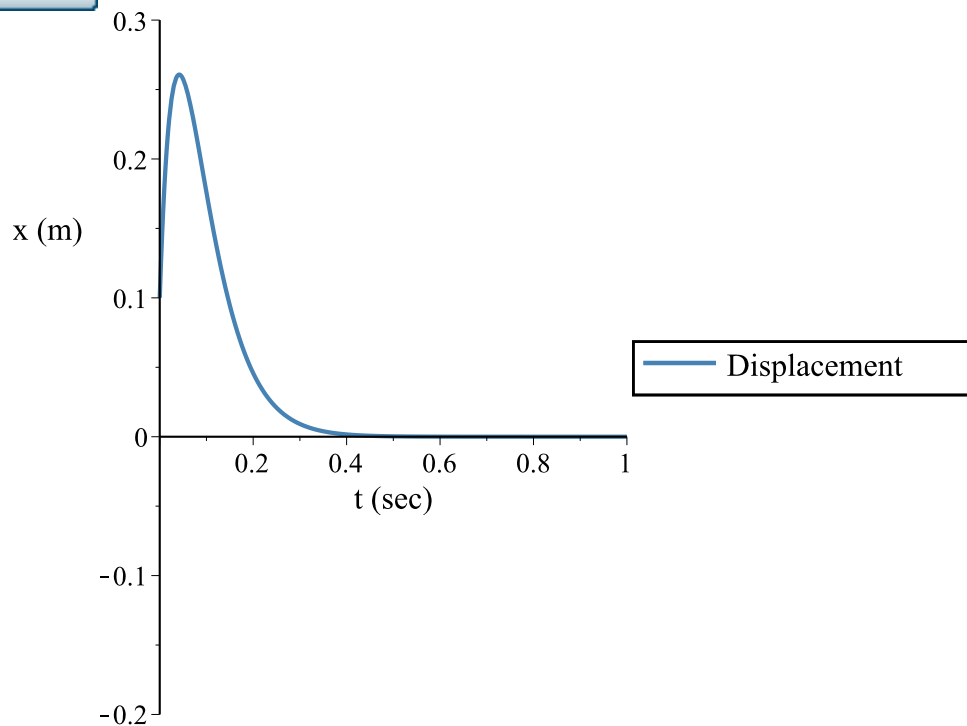
This case separates the oscillatory case from the nonoscillatory case. It is also the case that provides the fastest return to the equilibrium position without oscillation.

The following plot shows an overlay of the three plots for the different coefficients given above.

Overlay of Under, Over and Critically damped responses
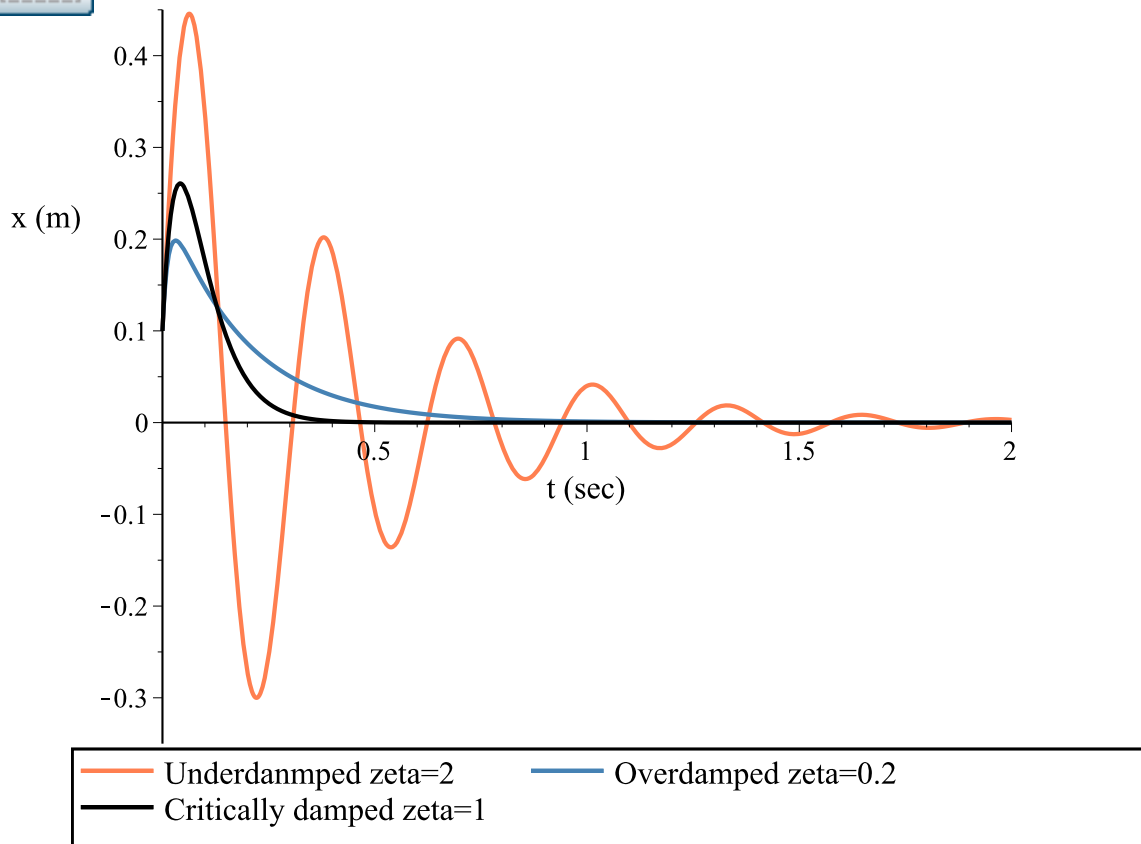
For the following plot, the damping ratio can be changed by changing the spring stiffness k and damping coefficient c using the gauges shown below. The initial conditions and parameters are the same as for the plot shown above, except for the spring stiffness and the damping coefficient which can be adjusted. $\left(m = 0.1\ kg,\ x_0 = 0.1\ m,\ v_0 = 10\ ms^{-1} \right)$

$c\ (1/$
$\quad m(N$
$\quad \cdot s))$

$k\ (N/$
$\quad\quad m)$

Displacement vs. time
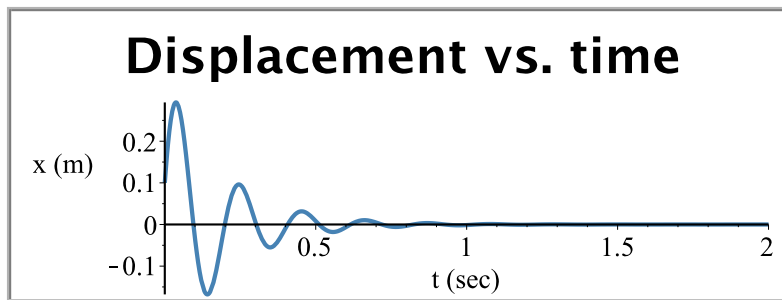
$$\zeta = \boxed{0.174}$$
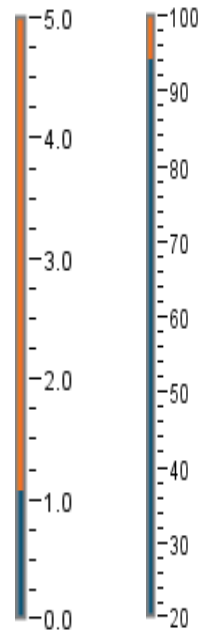
Examples with MapleSim

Example 1: Spring-Pendulum with damping

Problem Statement: A component of a machine is modeled as a pendulum connected to a spring (as shown on the right). Derive the equation of motion and find the natural frequency of the system. The mass of the pendulum is 2kg, the length of the pendulum is 0.5m and the stiffness of the spring is 20 N/m. Assume that the rod of the pendulum has no mass. The initial angular displacement is 0.175 rad (approx. 10°) measured from the vertical and there is no initial angular velocity. Find the approximate value of the damping coefficient required for the
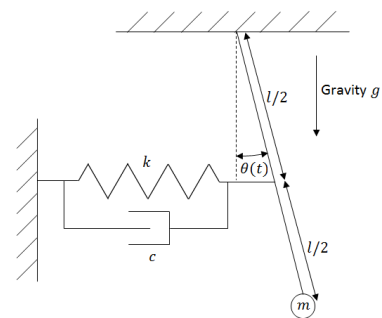
oscillations to die out after about 3 seconds.

▼ **Analytical Solution**

Data:

restart :

$m := 2 :$ [kg]

$l := 0.5 :$ [m]

$k := 20 :$ [N/m]

$g := 9.81 :$ [m/s^2]

$\theta_0 := 0.175 :$ [rad]

$\omega_0 := 0 :$ [rad/s]

Solution:

Since the angle is small, we will assume that the spring stretches and compresses in the horizontal direction only. Hence the force due to the spring can be written as

$$F_k := -k \cdot \left(\frac{l}{2} \right) \cdot \theta(t) :$$

and the force due to the damper can be written as

$$F_c := -c \cdot \left(\frac{l}{2} \right) \cdot \dot{\theta}(t) :$$

Taking the sum of the moments of force about the pivot we get

$$J_0 \cdot \ddot{\theta}(t) = F_k \cdot \left(\frac{l}{2} \right) + F_c \cdot \left(\frac{l}{2} \right) - m \cdot g \cdot l \cdot \theta(t) :$$

where J_0 is the moment of inertia of the pendulum. This equation can be rewritten as

$$m \cdot l^2 \cdot \ddot{\theta}(t) + c \cdot \left(\frac{l}{2}\right)^2 \cdot \dot{\theta}(t) + \left(k \cdot \left(\frac{l}{2}\right)^2 + m \cdot g \cdot l\right) \cdot \theta(t) = 0 :$$

or

$$m \cdot \ddot{\theta}(t) + \frac{c}{4} \cdot \dot{\theta}(t) + \left(\frac{k}{4} + \frac{m \cdot g}{l}\right) \cdot \theta(t) = 0 :$$

Comparing this equation to the form of Eq. (2) we can conclude that the natural frequency of the system is

$$\omega_n := \sqrt{\frac{\frac{k}{4} + \frac{m \cdot g}{l}}{m}} = 4.70$$

and the critical damping coefficient is

$$c_{cr} := 4 \cdot 2 \cdot m \cdot \omega_n = 75.25104651$$

From Eq. (12) it can be seen that, for underdamped motion, the amplitude of the oscillations decays with time and is equal to $A \cdot e^{-\zeta_d \cdot \omega_n \cdot t}$.

Here,

$$\zeta_d := \frac{c}{c_{cr}}$$

$$0.01328885174 \, c \qquad \qquad (2.1.1.1)$$

$$\omega_d := \omega_n \cdot \sqrt{1 - \zeta_d^2}$$

$$4.703190407 \sqrt{-0.0001765935806 \, c^2 + 1} \qquad (2.1.1.2)$$

and

$$A := \sqrt{\frac{\omega_d^2 \cdot \theta_0^2 + \left(\omega_0 + \zeta_d \cdot \omega_n \cdot \theta_0\right)^2}{\omega_d^2}}$$

$$\sqrt{\frac{-4.520795659 \; 10^{-15} \, c^2 + 0.03062499999}{-0.0001765935806 \, c^2 + 1}} \qquad (2.1.1.3)$$

Therefore the amplitude of the oscillations is given by

$$A \cdot \exp\left(-\zeta_d \cdot \omega_n \cdot t\right)$$

$$\sqrt{\frac{-4.520795659 \ 10^{-15} \ c^2 + 0.03062499999}{-0.0001765935806 \ c^2 + 1}} \ e^{-0.06250000002 \ c \ t} \tag{2.1.1.4}$$

Since this is never actually becomes 0, we need to define what we consider as "died-out" oscillations. For example in this case we can say that the oscillations have died out once the amplitude of the oscillations is less than 0.0035 rad (2 percent of the initial displacement).

$$A \cdot \exp\left(-\zeta_d \cdot \omega_n \cdot 3\right) = 0.0035$$

$$\sqrt{\frac{-4.520795659 \ 10^{-15} \ c^2 + 0.03062499999}{-0.0001765935806 \ c^2 + 1}} \ e^{-0.1875000001 \ c} = 0.0035 \tag{2.1.1.5}$$

use *RealDomain* **in** *solve*$\left(\sqrt{\frac{-4.520795659 \ 10^{-15} \ c^2 + 0.03062499999}{-0.0001765935806 \ c^2 + 1}} \ e^{-0.1875000001 \ c}\right.$

$\left. = 0.0035 \right)$ **end use**

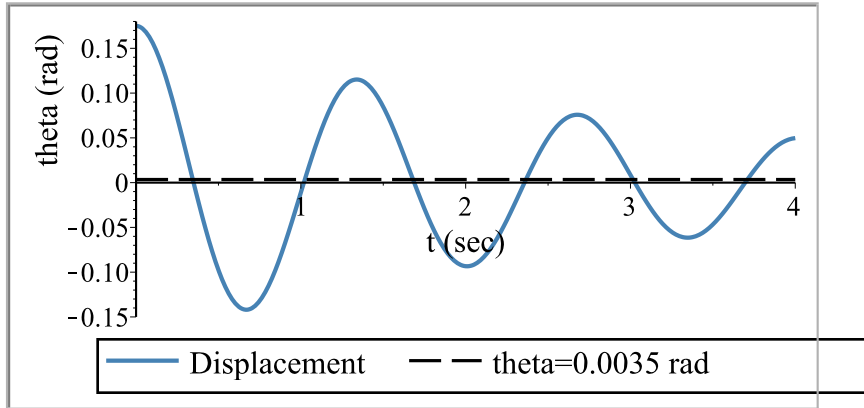$$21.08209418 \tag{2.1.1.6}$$

Therefore, the required damping coefficient is about 21 N·s/m.

▼ **Graphical solution**

An alternate method is to plot the displacement as a function of time and vary the damping constant until it appears to have died out at about 3 seconds. For the following plot, the slider can be used to adjust the damping coefficient to find an approximate value of the damping coefficient which causes the oscillations to die after 3 seconds.

Damping

Coefficie nt

theta (rad)

0.15
0.10
0.05
0
-0.05
-0.10
-0.15

1   2   3   4
t (sec)

—— Displacement   — — theta=0.0035 rad
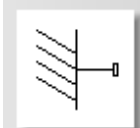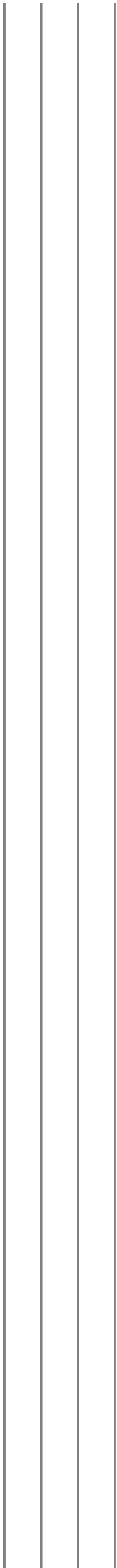
30
25
20
15
10
5

$c =$

5
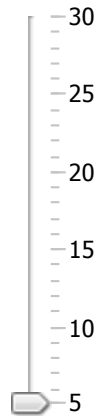
$$\left[\frac{N \cdot s}{m} \right]$$
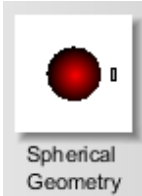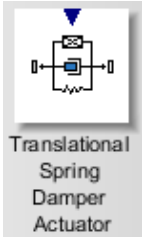
MapleSim Simulation

Constructing the model

Step1: Insert Components

Drag the following components into the workspace (these are the same ones used for the simulation in Part 1):

Table 1: Components and locations

Component	Location
Fixed Frame (2 required)	Multibody > Bodies and Frames
Revolute	Multibody > Joints and Motions

Rigid Body Frame (2 required)	Multibody > Bodies and Frames
Rigid Body	Multibody > Bodies and Frames
Spherical Geometry	Multibody > Visualization
Path Trace	Multibody > Visualization
Cylindrical Geometry	Multibody > Visualization
Translational Spring Damper Actuator	Multibody > Forces and Moments

 Spring Geometry	Multibody > Visualization
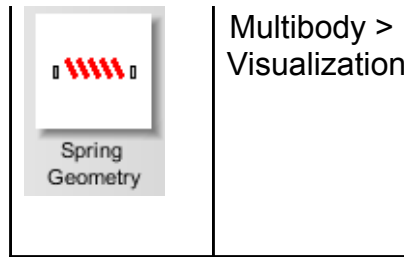
Step 2: Connect the components

Connect the components as shown in the following diagram (the dashed boxes are not part of the model, they have been drawn on top to help make it clear what the different components are for).
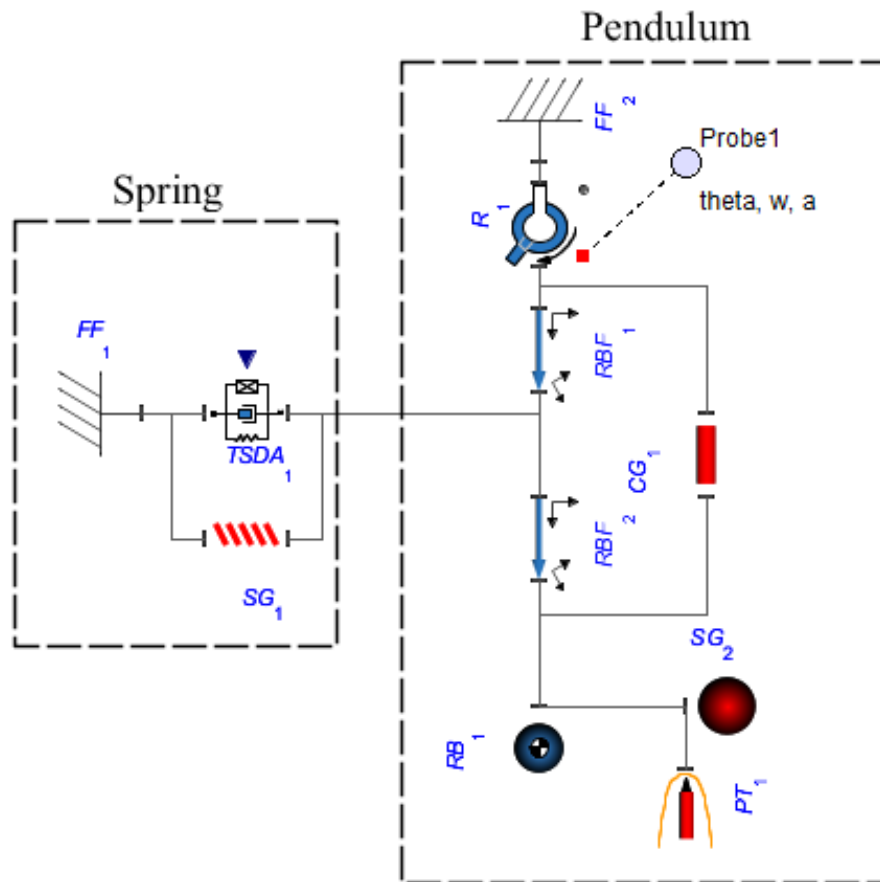
Fig. 3: Component diagram

Step 3: Set up the Pendulum

1. Click the **Revolute** component and enter **0.175** *rad* for the initial angle (θ_0) and select **Strictly Enforce** in the drop down menu for the initial conditions ($IC_{\theta,\,\omega}$). The axis of rotation ($\widehat{e}_1$) should be left as the default axis **[0,0,1]**.

2. Enter **[0,-0.25,0]** for the x,y,z offset ($\vec{r}_{XYZ}$) of both the **Rigid Body Frames**.

3. Enter **2** *kg* for the mass (*m*) of the **Rigid Body Frame**.

Step 4: Set up the Spring

1. Click the **Fixed Frame** component connected to the **TSDA** (**FF₁** in the

diagram) and enter **[-0.25,-0.25,0]** for the x,y,z offset ($\bar{r}_{XYZ}$).

2. Click the **TSDA** component and enter **20** *N/m* for the spring constant (K_{spring}). Also, enter **0.25** *m* for the unstretched length (l_0) to correspond to the location of the **Fixed Frame**. Enter an initial guess for the damping constant (K_{damper}). This damping constant value will be varied to find the value that satisfies the problem specifications.

Step 5: Set up the visualization (Inserting the **Visualization** components is optional)

1. Click the **Cylindrical Geometry** component and enter a value around **0.01** *m* for the radius.
2. Click the **Spherical Geometry** component and enter a value around **0.05** *m* for the radius.
3. Click the **Spring Geometry** component, enter a number around **10** for the number of windings, enter a value around **0.02** *m* for **radius1** and enter a value around **0.005** *m* for **radius2**.

Step 6: Run the Simulation

1. Click the **Probe** attached to the **Revolute** joint and select **Angle**, **Speed** and **Acceleration** to obtain plots of the angular position, speed and acceleration vs. time.
2. In the **Settings** tab, change the **Simulation duration** to around **4** *s* to make the plots that are generated easier to read.
3. Run the simulation.
4. Repeat the simulation by varying the value for the damping constant to find the value that satisfies the specifications.

The following video shows the visualization of the simulation.
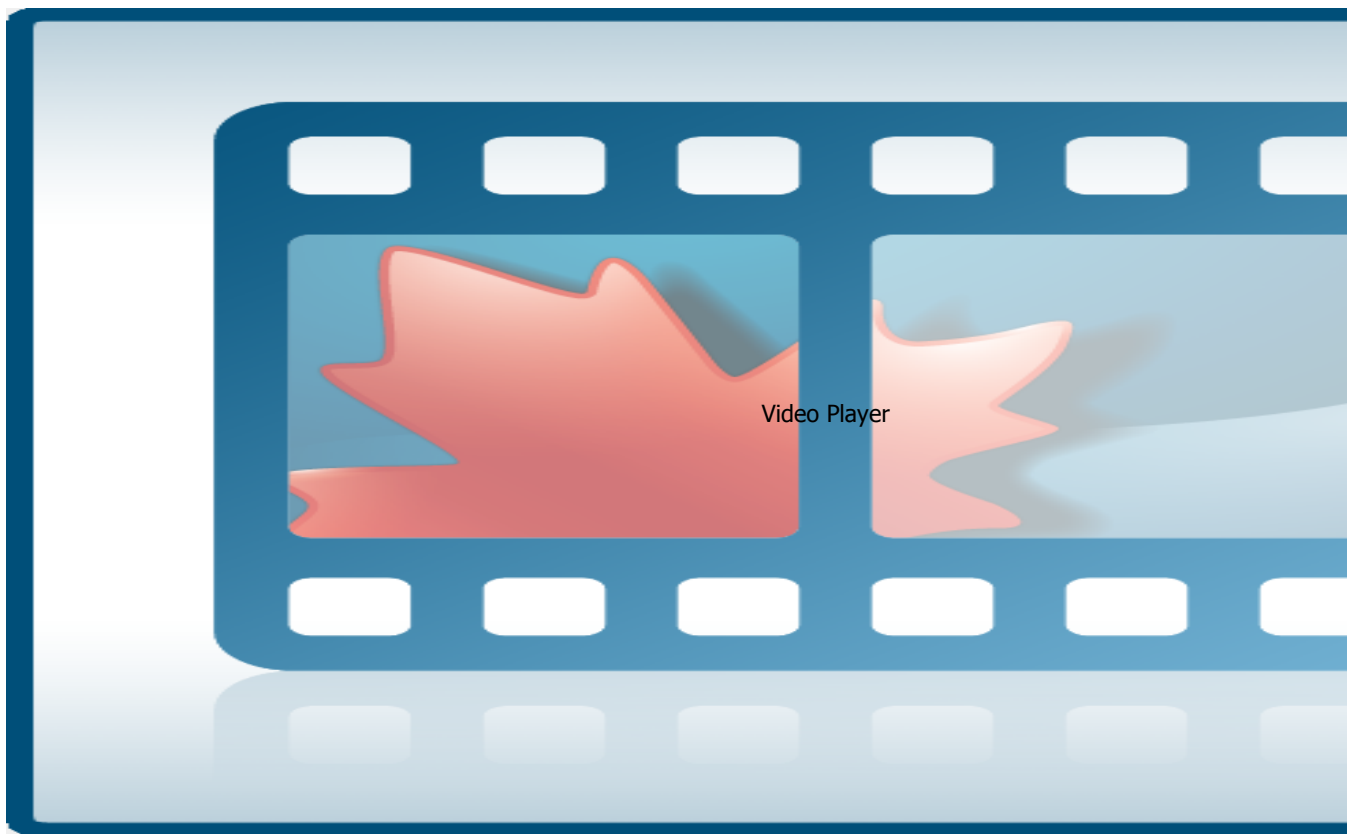
Video 1: Simulation visualization

▼ Example 2: Bicycle suspension

Problem Statement: The front wheel assembly of a bicycle with suspension can be crudely modeled as a mass, representing the tire and the wheel,attached to two parallel spring-dampers which represent the suspension. Fig. 4, on the right, illustrates this model. The mass m is approximately 3 kg.

If the stiffness of the springs that represents the suspension (k_1) is 50000 N/m, what should the minimum damping coefficient (c_1) be such that there are no

Fig. 4: Bicycle wheel model
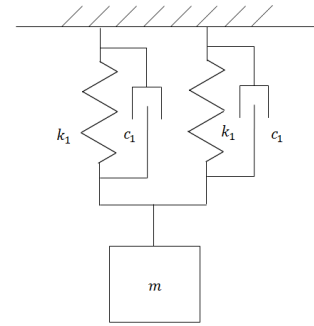
oscillations if the wheel is given an initial velocity of 1 m/s in a direction that compresses the assembly.

Analytical solution

Data:

$k_1 := 50000:$ [N/m]

$m := 3:$ [kg]

Solution:

The equation of motion for the mass can be written as

$m \cdot \ddot{y}(t) = -k_1 \cdot y(t) - k_1 \cdot y(t) - c_1 \cdot \dot{y}(t) - c_1 \cdot \dot{y}(t):$

or

$m \cdot \ddot{y}(t) + 2 \cdot k_1 \cdot y(t) + 2 \cdot c_1 \cdot \dot{y}(t):$

The minimum damping coefficient for which the system will not oscillate can be determined by finding the value of the damping coefficient for which the damping ratio is 1. Using Eq. (7) we get the following equation which can be solved to find the required damping coefficient.

$$1 = \frac{2 \cdot c_1}{2 \cdot \sqrt{2 \cdot k_1 \cdot m}} \; :$$

$$solve\left(1 = \frac{2 \cdot c_1}{2 \cdot \sqrt{2 \cdot k_1 \cdot m}}\right) = 100\sqrt{30} \xrightarrow{\text{at 5 digits}} 547.72$$
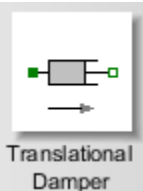
Therefore the damping coefficient c_1 should be approximately 550 [N·s/m] for there to be no oscillations.

MapleSim Simulation

Step1: Insert Components

Drag the following components into the workspace:

Table 2: Components and locations

Component	Location
Translational Fixed	1-D Mechanical > Translational > Common
Translational Spring (2 required)	1-D Mechanical > Translational > Common
Translational Damper (2 required)	1-D Mechanical > Translational > Common
	1-D Mechanical >

Mass	Translational > Common
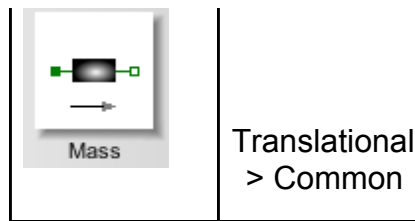
Step 2: Connect the components

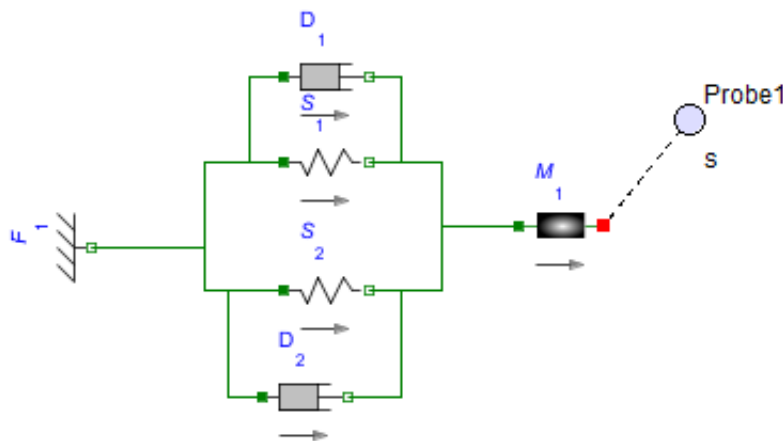Connect the components as shown in the following diagram.

Fig. 13: Component diagram

Step 3: Set parameters and initial condition

1. Enter **50,000** *N/m* for the spring constants (*c*) of both the **Translational Spring** components.
2. Enter an initial guess (for example **100** *N·s/m*) for the damping constants (*d*) of both the **Translational Damper** components.
3. Click the **Mass** component, enter **3** *kg* for the mass (*m*) and **-1** *m/s* for the initial velocity (v_0). Select the check mark that enforces this initial condition.

Step 4: Run the Simulation

1. Attach a **Probe** to the **Mass** component as shown in Fig. 13. Click this probe and select **Length** in the **Inspector** tab. This will show the position of the mass as a function of time.

2. Click **Run Simulation** (▶).

3. Rerun the simulation by varying the damping coefficient using a systematic trial and error approach to find the approximate value of the damping coefficient that leads to no oscillations.

Reference:

D. J. Inman. "Engineering Vibration", 3rd Edition. Upper Saddle River, NJ, 2008, Pearson Education, Inc.