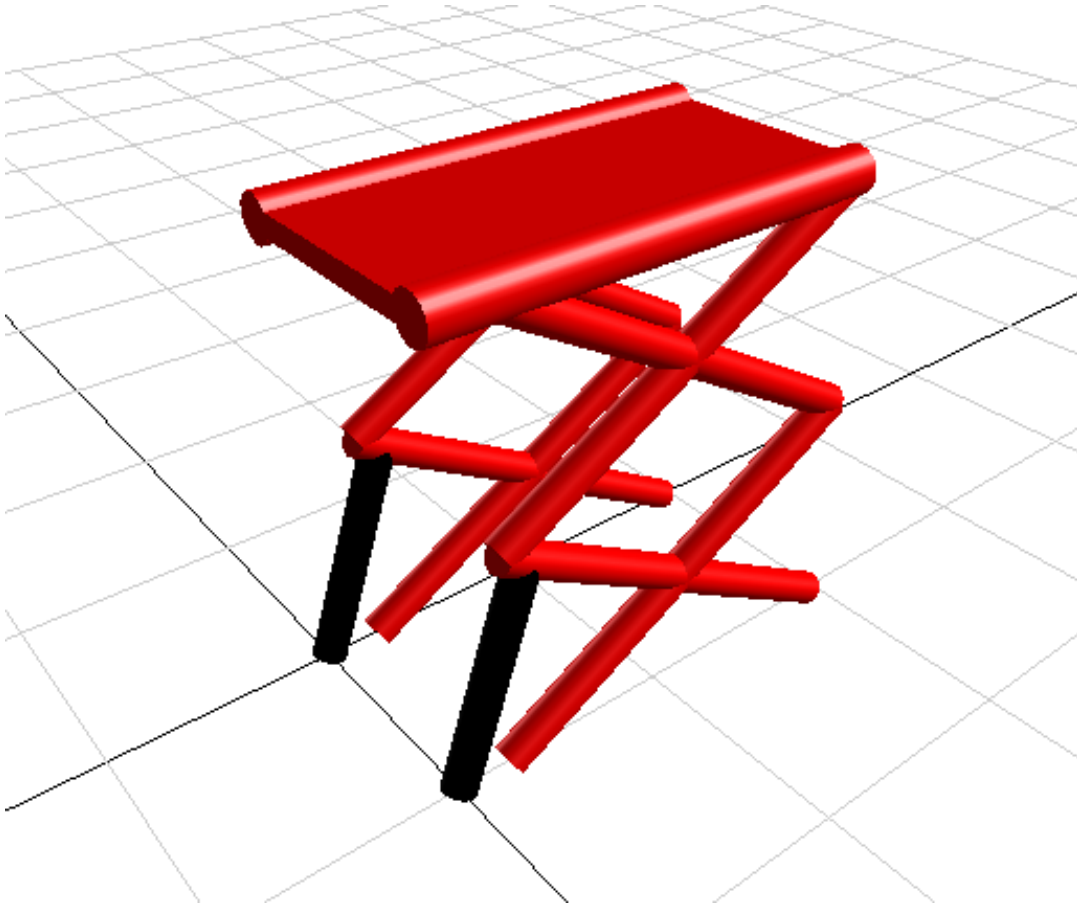


# Work and Energy



The motion of a particle can be examined with the method of work and energy. Although many problems can be solved through Newton's second law, work and energy are very useful tools which can often make a solution easier to find. This module will discuss many aspects of work such as the work done by a weight, springs and gravity. We will also discuss kinetic and potential energy, the conservation of energy, and the relationships between power, energy and work. These topics are all related to each other and help to solve many dynamic applications.



The concept of work relates directly to the force, mass, the displacement of the object and the velocity it is travelling at. An object experiences work if it is displaced while under an applied force. It is a scalar quantity with a sign, but no direction. Work is measured as  $ft \cdot lb$  or  $in \cdot lb$  in U.S. customary units, and  $N \cdot m$ , also known as Joules  $J$ , in S.I. units. As defined, the work a force exerts is defined as

$$dU = F \cdot dr$$

... Eq. (1)

where  $F$  is the force acting on the object and  $dr$  is the differential displacement the object has travelled. The vectors  $F$  and  $dr$  are multiplied through dot product to find the scalar value of work.

$$dU = |F| \cdot |dr| \cdot \cos(\alpha)$$

... Eq. (2)

Work can also be expressed as the sum of the product of each component.

$$dU = F_x \cdot dx + F_y \cdot dy + F_z \cdot dz$$

... Eq. (3)

A finite displacement of a force can be expressed as the integral of Eq. (1) across the displacement of the force.

$$U_{1 \rightarrow 2} = \int_{A_1}^{A_2} F \cdot dr$$

... Eq. (4)

If this is expressed in the rectangular components,

$$U_{1 \rightarrow 2} = \int_{A_1}^{A_2} (F_x \cdot dx + F_y \cdot dy + F_z \cdot dz)$$

... Eq. (5)

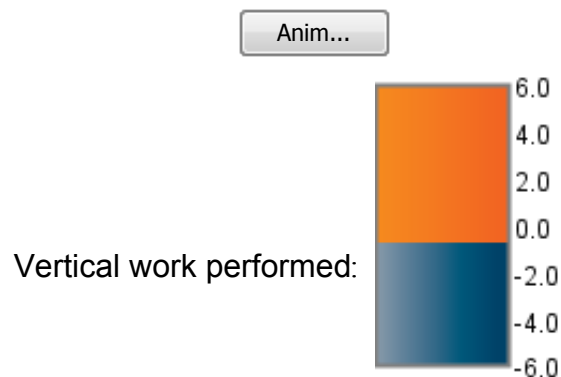
In rectilinear motion (where a particle will travel in a straight line), work can be found simply as the product of the component of force acting along the line of motion and the displacement of the particle.

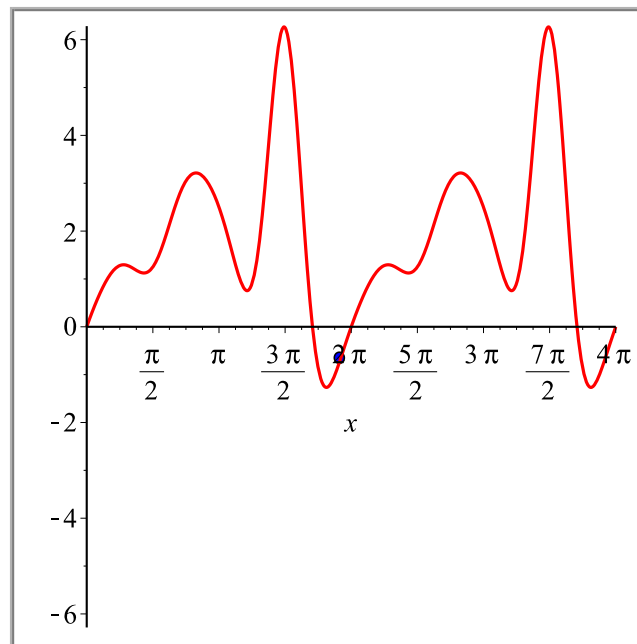
$$U_{1 \rightarrow 2} = F \cdot \cos(\alpha) \cdot \Delta x$$

... Eq. (6)

### Example of Work

In the plot below, a blue dot represents a car travelling along a path. The vertical work done by the car is displayed with the gauge as the 'car' goes over the hills. The work is absolute, so when the car returns to its original height it has performed no work.





## Work Done by Gravity

Gravity exerts work on objects as well. This work can be expressed similarly to the previous definitions. The force acting on an object under gravity is  $W = m \cdot g$ , and the displacement an object falls is  $dy$ . Work is expressed as

$$dU = -W \cdot dy$$

... Eq. (7)

The work is negative because the force of gravity acts in the downward direction. This means that if a body is falling, positive work is being done by gravity. Conversely, if an object rises, negative work is performed. This can be understood by recognizing that it takes energy to move two bodies away from each other if they are attracted by gravity. Integrating the work from height  $y_1$  to  $y_2$  and simplifying,

$$U_{1 \rightarrow 2} = -W \cdot \Delta y$$

... Eq. (8)

In cases where gravitational force is considered, the force is modelled by

$$F = \frac{G \cdot M \cdot m}{r^2}$$

... Eq. (9)

Gravitational force attracts two objects. If the distance is decreasing as work is being done, the work is negative.

$$dU = -F dr = -\frac{G \cdot M \cdot m}{r^2} dr$$

... Eq. (10)

During a finite displacement, the work done will be

$$U_{1 \rightarrow 2} = -\int_{r_1}^{r_2} \frac{G \cdot M \cdot m}{r^2} dr = G \cdot M \cdot m (r_2 - r_1)$$

... Eq. (11)

This equation is often used to measure the work done by objects in orbit.

## Work of a Spring

Springs exert force on an object as well and therefore perform work. A spring will not exert a force on a body if it is undeformed. It will exert a force as the spring lengthens or compresses. This force is modelled by

$$F = k \cdot x$$

... Eq. (12)

This force can model work. A spring will apply work on a body as the spring returns to its undeformed state, so the work will be negative.

$$dU = -F dx$$

... Eq. (13)

Substituting in the force of a spring in Eq. (9) and integrating from  $x_1$  to  $x_2$ ,

$$U_{1 \rightarrow 2} = - \int_{x_1}^{x_2} k \cdot x \, dx$$

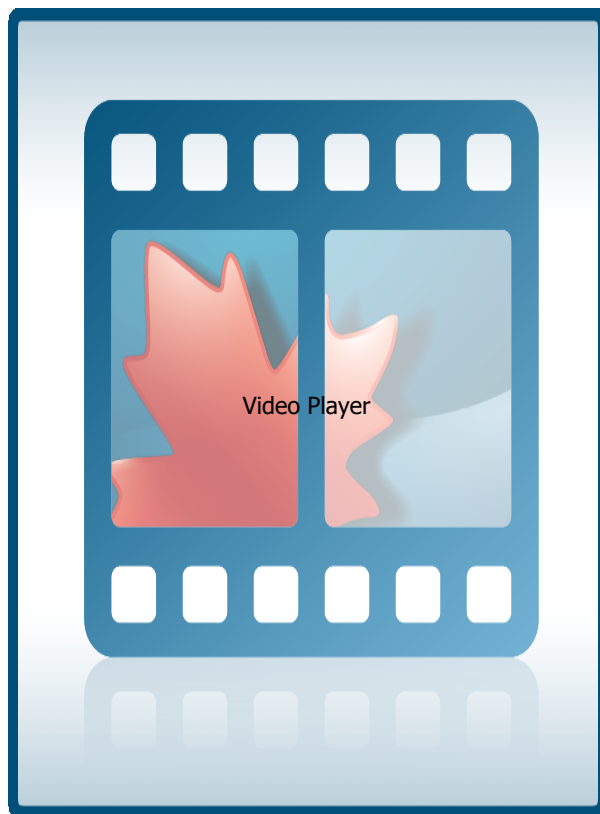
... Eq. (14)

Which evaluates and simplifies to

$$U_{1 \rightarrow 2} = \frac{1}{2} \cdot k \cdot (x_1^2 - x_2^2)$$

... Eq. (15)

This will be the total work a spring performs on a body as it deforms.



Video 1: Spring stops a ball

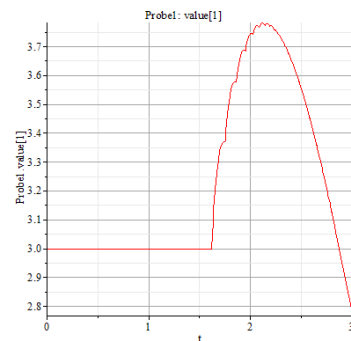


Figure 1: Spring Response

In this video, a pendulum swings from horizontal and hits a ball. The blue ball rolls into

the spring, which retracts and performs work to stop the ball from rolling. The position of the tip of the spring can be seen in the graph on the right. The spring compressed by about 0.85 m upon impact from the ball.

## Energy

Energy comes in many forms. It can be stored or used and can change forms. The law of the conservation of energy states that energy is never lost nor gained. As two objects collide, energy may seem to be lost in the impact. However, the energy lost merely changes from kinetic or potential energy to another form such as sound, heat or light. The net energy in any system is always conserved. For most applications, assuming no energy is lost is accurate enough to solve the problem. Many applications exist which do not conserve energy but that will not be focus of this section.

## Kinetic Energy

A moving object has energy.

Consider a particle moving in a rectilinear or curved path. A force acting on this particle can be shown with Newton's second law  $F = m \cdot a$ . Knowing that acceleration is the derivative of velocity,

$$F = m \cdot \frac{dv}{dt}$$

... Eq. (16)

By multiplying by  $\frac{ds}{ds}$  we can obtain

$$F = m \cdot \frac{dv}{ds} \cdot \frac{ds}{dt}$$

... Eq. (17)

Knowing that  $\frac{ds}{dt}$  is velocity, and moving ds to the left hand side,

$$F \cdot ds = m \cdot v \cdot dv$$

... Eq. (18)

Integrating this reveals

$$\int_{s_1}^{s_2} F \, ds = m \cdot \int_{v_1}^{v_2} v \, dv = \frac{1}{2} \cdot m \cdot v_2^2 - \frac{1}{2} \cdot m \cdot v_1^2$$

... Eq. (19)

Finally, the left side of the equation can be seen to represent work. The right hand side is the change in energy. This is the principle of work and energy. It is understood that the work of a force is equal to the change in energy of the particle. The expression on the right of the equation is the kinetic energy of a particle. Explicitly this is defined as

$$T = \frac{1}{2} \cdot m \cdot v^2$$

... Eq. (20)

which is the kinetic energy at an instantaneous moment of time. This combined with Eq. (19) gives the expression for the principle of work and energy.

$$T_1 + U_{1 \rightarrow 2} = T_2$$

... Eq. (21)

As work is applied, an objects kinetic energy will change by the magnitude of that work.

## ▼ Potential Energy

Another form of energy is potential energy. This form of energy is stored and ready to be used. A battery is a good example of potential energy because it stores power and releases energy until it is depleted, but it does not move. Potential energy can be related to the opposite of kinetic energy. An object loses potential energy as work is applied to it.



Refer to Eq. (8) about the work done by gravity where  $W$  is an objects weight under gravity  $W = m \cdot g$ . The change in height  $y$  is evaluated explicitly below.

$$U_{1 \rightarrow 2} = W \cdot y_1 - W \cdot y_2$$

... Eq. (22)

The potential energy with respect to the force of gravity is denoted by  $V_g = W \cdot y$ . It is also commonly used in the alternate form

$$V_g = m \cdot g \cdot h$$

... Eq. (23)

where  $h$  is the objects height above the ground, assuming it cannot go any lower. The general relation between work and potential energy can be seen as

$$V_{g_1} - U_{1 \rightarrow 2} = V_{g_2}$$

... Eq. (24)

This is only valid of course if the weight  $W$  remains constant. An object orbiting the earth may have a different force of gravity acting on it as it enters or leaves the earths orbit. Considering the force of gravity, Eq. (22) re-evaluates to

$$U_{1 \rightarrow 2} = \frac{G \cdot M \cdot m}{r_2} - \frac{G \cdot M \cdot m}{r_1}$$

... Eq. (25)

The expression that should be used if we cannot neglect the change of the force of gravity then becomes

$$V_g = - \frac{G \cdot M \cdot m}{r}$$

... Eq. (26)

Potential energy exists in multiple forms as well. In springs, potential energy is the energy the spring can exert on the object. A spring will only exert a force if it is deformed. Refer to the work of a spring in Eq. (15). The potential energy a spring has is represented by

$$V_e = \frac{1}{2} \cdot k \cdot x^2$$

... Eq. (27)

It is important to note that, similar to gravity, as work is applied, potential energy decreases. The spring is expending energy via the work it performs. The elastic potential energy and work of a spring is equivalent to that of gravity in Eq. (24).

## Conservation of Energy

Kinetic and potential energy relate to each other through the conservation of energy. This law states that no energy is ever lost or gained; it is always conserved. Adding up all forms of kinetic and potential energy in a system will produce the total energy in the system which is conserved. Isolate the work  $U_{1 \rightarrow 2}$  in Eq. (21) and Eq. (24) and sum up the energy.

$$V_1 - V_2 = T_2 - T_1$$

... Eq. (28)

This is rewritten as the sum of energy before and then after work is performed.

$$T_1 + V_1 = T_2 + V_2$$

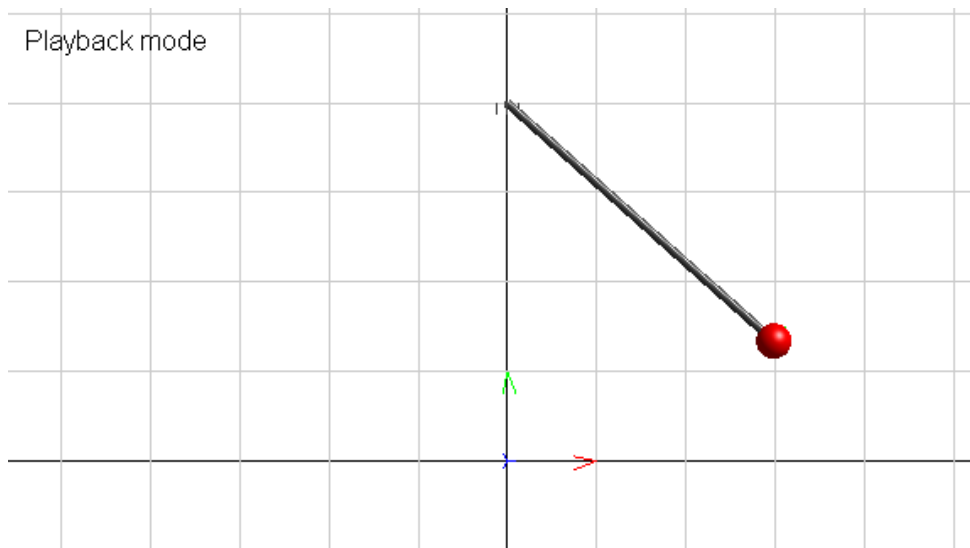
... Eq. (29)

This shows that all energy in a mechanical system, the **total mechanical energy**, must be conserved. This holds true for other forms of energy too. If friction were considered, energy would be lost in the form of heat. The work performed by the friction converts mass into thermal energy. Thus, the sum of thermal and mechanical energy would

remain constant through the system. Other forms of energy may take or give energy to a system and should be considered when appropriate, such as electrical energy or chemical energy. For example, a battery can convert potential chemical energy into electrical energy. This could drive a motor which would gain kinetic mechanical energy and lose some heat from friction. However, the total energy of the system still remains a constant value.

### ▼ Energy in a Pendulum

Below is a simple pendulum modelled in MapleSim. Probes were attached to measure the potential and kinetic energy of the ball at the end of the pendulum. A third probe was attached to plot the sum or total energy of the system. The results are shown in figure 2.



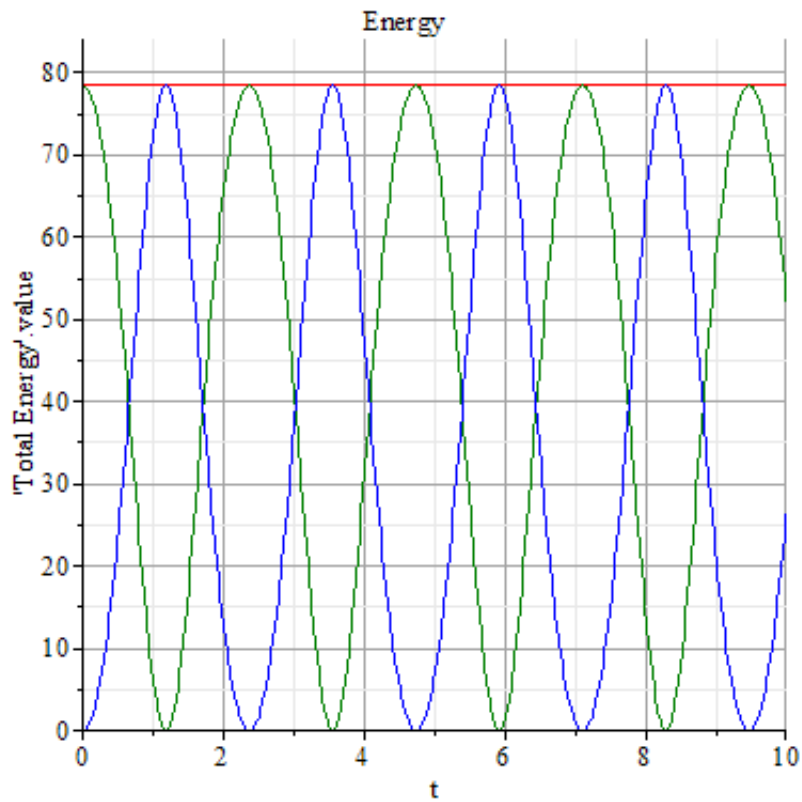


Figure 2: Pendulum and Energy plot

The results in the plot display the total energy (red), kinetic energy (blue) and potential energy (green) in the system. The sum of energy in the system remains consistent. The system modelled below is the same as figure 2 except that a damper was added that removes energy from the pendulums swings based on its velocity.

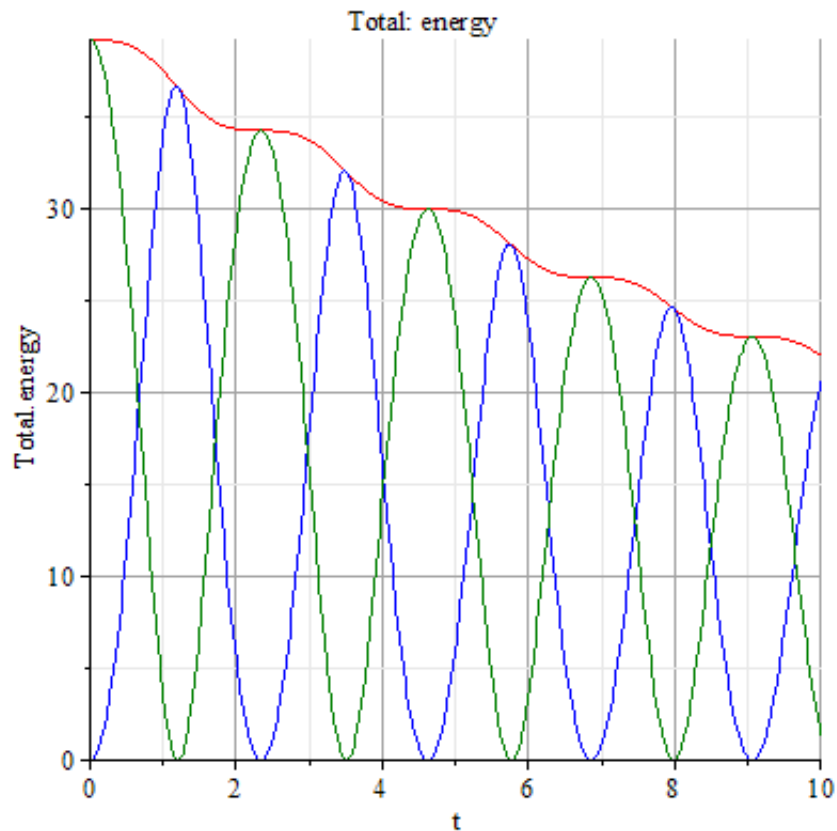


Figure 3: Damped Pendulum and Energy plot

The sum of total energy remains consistent in this system as it should. Total energy is decreasing over time due to the damper; it is not decreasing linearly because the pendulum is not always moving so the damper is not continuously affecting the system. As the pendulum slows to a stop at the top of each swing, the damper does not affect the system and the energy in the system flattens out and remains consistent.

## Examples with MapleSim

### Example 1: Braking Vehicle

**Problem Statement:** A 1300 kg car initially travelling at 50 km/h drives toward a raised bridge. If the car begins to apply its brakes 20 meters away from the edge of the bridge,

how much force needs to be constantly applied to the car by the brakes to stop it safely?

### Analytical Solution

*restart;*

#### Data:

$$v_0 := 50 \frac{[[km]]}{[[h]]} :$$

$$v_1 := 0 :$$

$$d := 20[[m]] :$$

$$m_{car} := 1300[[kg]] :$$

#### Solution:

We will need to use Eq. 21 to solve this problem. First convert  $v_0$  into SI units.

$$v_0 := 50 \cdot \left( \frac{1000[[m]]}{3600[[s]]} \right) = \frac{125}{9} \frac{[[m]]}{[[s]]}$$

Given the initial velocity and mass, we can calculate the initial kinetic energy in the system.

$$T_0 := \frac{1}{2} \cdot m_{car} \cdot v_0^2 = 125385.8 [[J]]$$

There is no drop in height because the car does not go over the edge, so there is no potential energy in this case.

After the car stops, the kinetic energy will be 0 with no potential energy. From Eq. 21 we know

$$T_0 + U_{1 \rightarrow 2} = 0 :$$

All of the force is acting in the same direction as the motion of the car.

$$U_{1 \rightarrow 2} = F \cdot d \cdot \cos(0) :$$

$$U_{1 \rightarrow 2} := F \cdot d :$$

Substituting all values into Eq. 21 reveals

$$\frac{1}{2} \cdot m_{car} \cdot v_0^2 + F \cdot d = 0 :$$

Solving and evaluating for F shows that the applied force must be at least

$$F := -\frac{1}{2} \cdot m_{car} \cdot v_0^2 \cdot \frac{1}{d} = -6269.3 \text{ [N]}$$

This can be confirmed by evaluating Eq. 21 from earlier.

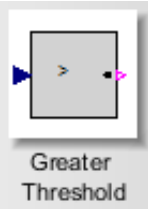
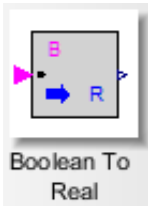
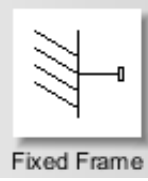

$$T_0 + U_{1 \rightarrow 2} = 0$$

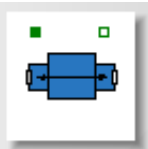
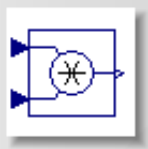
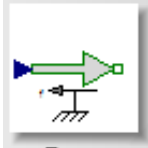
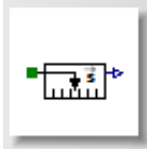
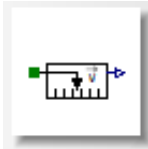
The system does finish with zero energy, so the car has stopped moving within 20 meters of when it started braking.

## MapleSim Solution

### Step 1: Insert Components

Drag the following components into a new workspace.

Component	Location
 <p>Greater Threshold</p>	Signal Blocks > Relational
 <p>Boolean To Real</p>	Signal Blocks > Signal Converters
 <p>Fixed Frame</p>	Multibody > Bodies and Frames
 <p>Rigid Body</p>	Multibody > Bodies and Frames

 <p>Prismatic</p>	<p>Multibody &gt; Joints and Motions</p>
 <p>Product</p>	<p>Signal Blocks &gt; Mathematical &gt; Operators</p>
 <p>Force</p>	<p>1D Mechanical &gt; Translational &gt; Force Drivers</p>
 <p>Position Sensor</p>	<p>1D Mechanical &gt; Translational &gt; Sensors</p>
 <p>Speed Sensor</p>	<p>1D Mechanical &gt; Translational &gt; Sensors</p>

**Step 2:** Connect the components.

Connect the components as shown in the diagram below.



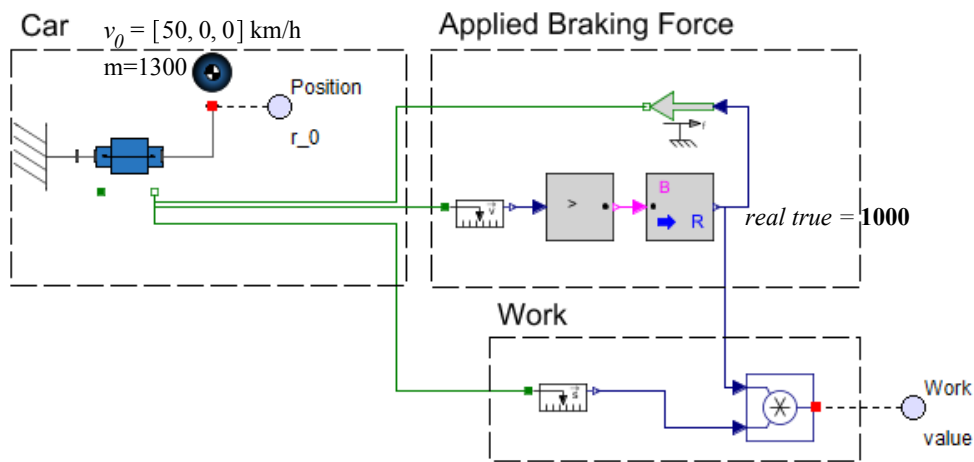



Figure 4: Braking car diagram

### Step 3: Establish parameters

1. On the main worksheet, select the **parameters**  icon and in the first table, set a new variable **F** at **-1000**. This will be the magnitude of the force, in Newtons, acting against the car.
2. Select the **Boolean to Real** block and on the 'Inspector' tab, set the 'real true' value to the variable **F**.
3. Select the **Rigid Body** and on the 'Inspector' tab, set the *mass* parameter to 1300 and confirm units are in kilograms. Also set the  $IC_{r,v}$  initial conditions to **Strictly Enforce** and change  $v_0$  to [50, 0, 0] in *km/h* units.

### Step 4: Run simulation.

Run the simulation and examine the generated plot. Change the values of F on the parameters page to observe the effects of a stronger or weaker force.

By observation, a force of 6300 N will stop the car 19.8 m from where it began applying the brakes.

## Results

The simulation will generate the following results.

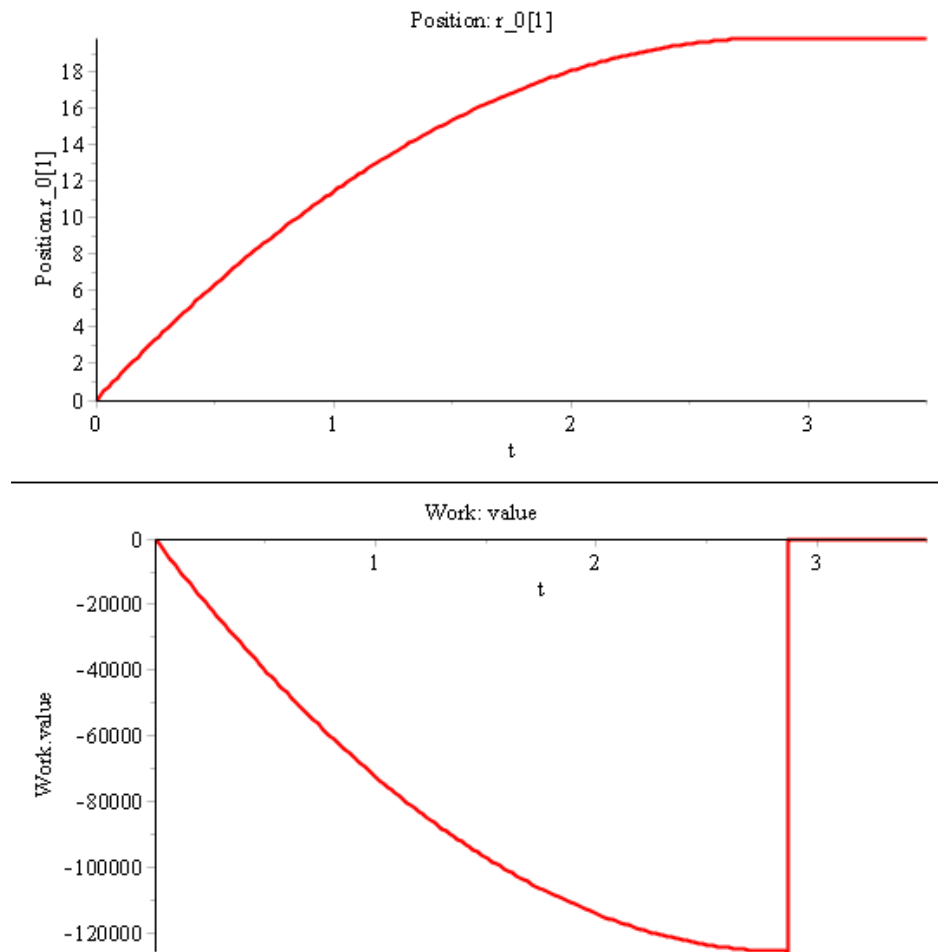


Figure 5: Results of the simulation.

## Example 2: Spring and Collar

**Problem Statement:** a 600 gram collar can slide along a horizontal, semicircular

rod ABD. The collar is attached to the spring CE, which has a spring constant of 135 N/m and an undeformed length of 250 mm. Determine the speed of the collar around the rod at B and D if the collar is released from rest at A. Neglect any friction.

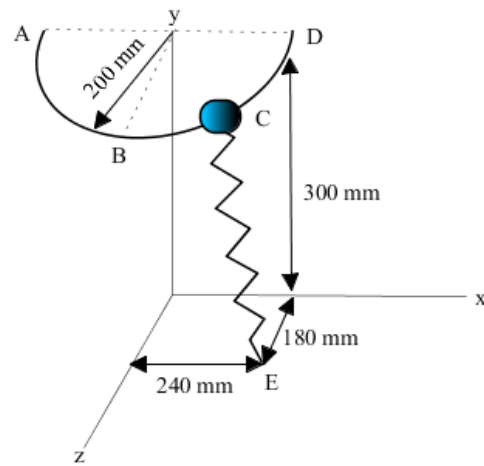


Figure 6: Spring and collar system

### Analytical Solution

*restart;*

**Data:**

$$m_c := 0.6 :$$

$$K_s := 135 :$$

$$S_s := 0.250 :$$

**Solution:**

First, establish the equations of motion for the collar.

$$x_c := -0.200 \cdot \cos(\text{Pi} \cdot t) :$$

$$y_c := 0.300 :$$

$$z_c := 0.200 \cdot \sin(\text{Pi} \cdot t) :$$

where  $0 \leq t \leq 1$ . The collar will be at A when  $t = 0$ , B when  $t = 0.5$  and D when  $t = 1$ .

Next, find the deformation  $\Delta x_s$  of the spring based on the position of the collar.

Calculate this by finding the length of the spring and subtracting the undeformed

length  $S_s$ .

$$\Delta x_s := \sqrt{(0.240 - x_c)^2 + (0.300)^2 + (0.180 - z_c)^2} - S_s :$$

Recall the energy stored in a spring, Eq. 27. Calculate the total potential energy stored in the spring at each A, B and D.

$$V_{s_A} := \frac{1}{2} \cdot K_s \cdot (\Delta x_s)^2$$

$$\frac{135}{2} \left( \sqrt{(0.240 + 0.200 \cos(\pi t))^2 + 0.090000 + (0.180 - 0.200 \sin(\pi t))^2} - 0.250 \right)^2 \quad (3.2.1.1)$$

Evaluate  $V_{s_A}$  at  $t=0$   $\xrightarrow{\text{evaluate at point}}$

$$6.576566415 \quad (3.2.1.2)$$

$$V_{s_B} := \frac{1}{2} \cdot K_s \cdot (\Delta x_s)^2$$

$$\frac{135}{2} \left( \sqrt{(0.240 + 0.200 \cos(\pi t))^2 + 0.090000 + (0.180 - 0.200 \sin(\pi t))^2} - 0.250 \right)^2 \quad (3.2.1.3)$$

Evaluate  $V_{s_B}$  at  $t=0.5$   $\xrightarrow{\text{evaluate at point}}$

$$1.224865757 \quad (3.2.1.4)$$

$$V_{s_D} := \frac{1}{2} \cdot K_s \cdot (\Delta x_s)^2$$

$$\frac{135}{2} \left( \sqrt{(0.240 + 0.200 \cos(\pi t))^2 + 0.090000 + (0.180 - 0.200 \sin(\pi t))^2} - 0.250 \right)^2 \quad (3.2.1.5)$$

Evaluate  $V_{s_D}$  at  $t=1$   $\xrightarrow{\text{evaluate at point}}$

$$0.7041486182 \quad (3.2.1.6)$$

Using the conservation of energy Eq. 29, the velocities can be found. No energy is

lost and there is no initial kinetic energy when the collar is at A.

$$V_{s_A} + 0 = V_{s_B} + T_{c_B}$$

Substituting in the values for energy that were calculated,

$$eq1 := (3.2.1.2) = (3.2.1.4) + \frac{1}{2} \cdot m_c \cdot v_B^2 :$$

$$solve(eq1, v_B) = -4.223624296, 4.223624296$$

We know that the calculated value is simply a magnitude of the velocity, so select the positive value.

$$v_B = 4.223624296$$

The velocity at D can be found the same way as  $v_B$ .

$$V_{s_A} + 0 = V_{s_D} + T_{s_D}$$

Substitute the calculated values from earlier and solve for  $v_D$ .

$$eq2 := (3.2.1.2) = (3.2.1.6) + \frac{1}{2} \cdot m_c \cdot v_D^2 :$$

$$solve(eq2, v_D) = -4.424333395, 4.424333395$$

Again, ignore the negative value.

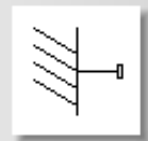

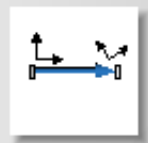

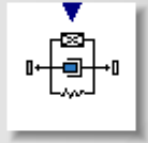
$$v_D = 4.424333395$$

The speed of the collar at point B is 4.22 seconds, and at point D is 4.42 seconds.

### MapleSim Solution

**Step 1:** Insert Components

Insert the following components into a new worksheet.

Component	Location
 Fixed Frame  (2 required)	Multibody > Bodies and Frames
 Rigid Body	Multibody > Bodies and Frames
 Rigid Body Frame	Multibody > Bodies and Frames
 Revolute	Multibody > Joints and Motions
 Translational Spring Damper Actuator	Multibody > Forces and Moments

**Step 2:** Connect the Components

Connect the components as show in the diagram below. The red box shows the components that create the semicircular disk and collar. The blue rectangle represents the spring.

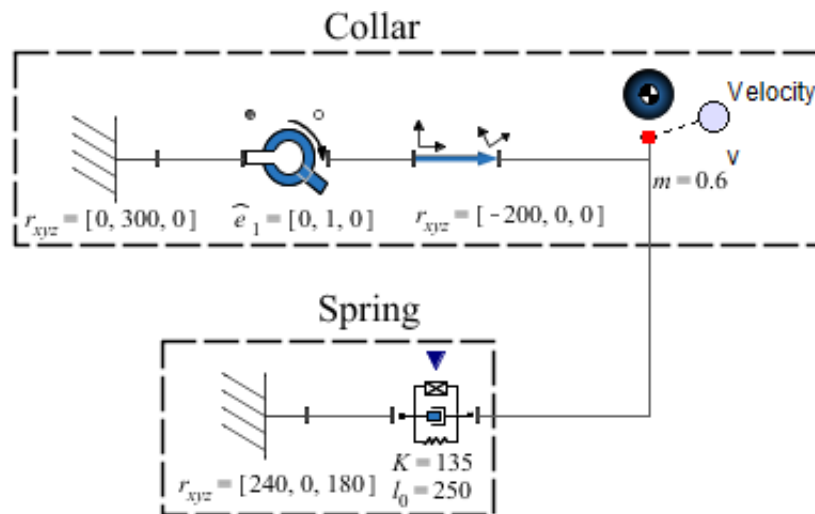



Figure 7: MapleSim diagram of spring and collar

### Step 3: Set Up Semicircular Rod.

1. Select the **Fixed Frame**. On the 'Inspector' tab, set the  $\bar{T}_{xyz}$  parameter to **[0, 300, 0]** and set the units to **mm**.
2. Select the **Revolute** block. On the 'Inspector' tab, set the  $\hat{e}_1$  parameter to **[0, 1, 0]** so that it will rotate about the y-axis.
3. Select the **Rigid Body Frame** block. On the 'Inspector' tab, set the  $\bar{T}_{xyz}$  parameter to **[-200, 0, 0]** and set the units to **mm**. This sets the starting location of the collar at point A.
4. Select the **Rigid Body**. On the 'Inspector' tab, set the mass  $m$  parameter to **0.6 kg**. Attach a probe  to the **Rigid Body** and on the 'Inspector' tab, select the **[1]** and **[3]** boxes under 'Speed'. This will measure the absolute x and z velocities, respectively, of the collar.

#### Step 4: Set Up Spring

1. Select the **Translational Spring Damper Actuator**. On the 'Inspector' tab, set the spring constant  $K_{Spring}$  parameter to **135** N/m. Also, set the unstretched length  $l_0$  parameter to **250** and change the units to **mm**.
2. Select the **Fixed Frame**. Set the  $\bar{r}_{xyz}$  parameter to **[240, 0, 180]** on the 'Inspector' tab, and set the units to **mm**.

#### Step 5: Run Simulation

Run the simulation and examine the plots generated by the probe. You may want to shorten the time of the simulation. Under the 'Settings' tab, set the simulation time  $t_d$  to 2 seconds.

#### ▼ Results

The simulation generates the following results.



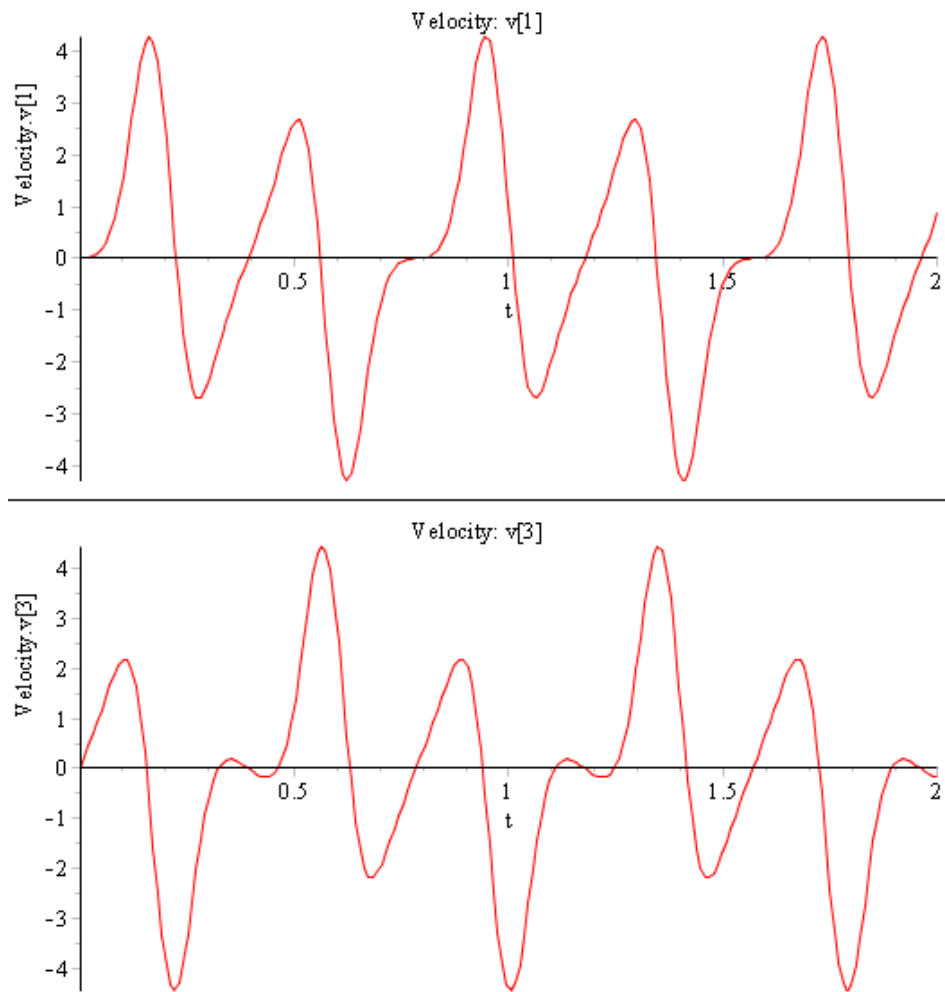
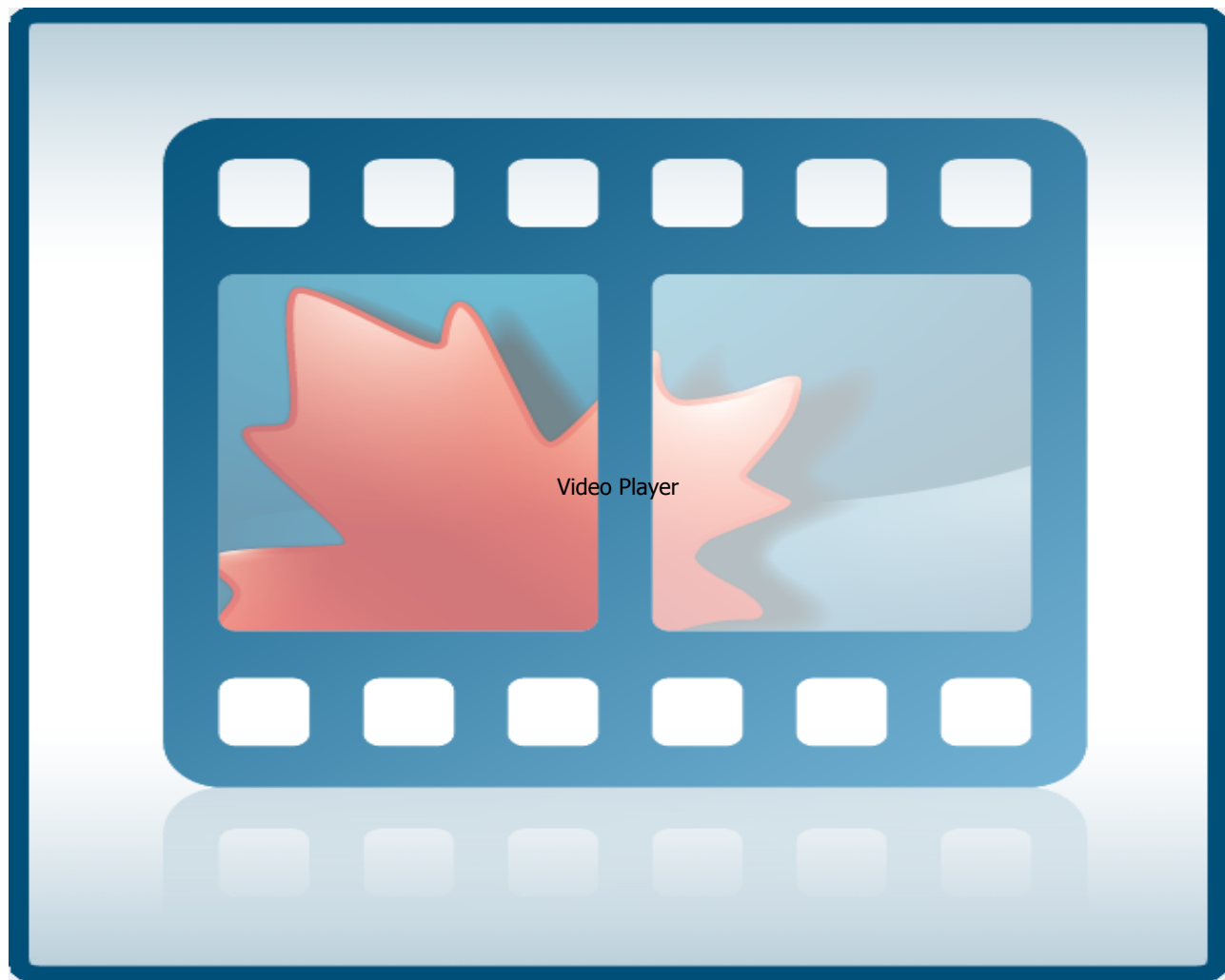


Figure 8: Simulation results

The following video is a demonstration of the model created in MapleSim.



Video 2: Spring and collar simulated in MapleSim

References:

Beer et al. "Vector Mechanics for Engineers: Dynamics", 10th Edition. 1221 Avenue of the Americas, New York, NY, 2013, McGraw-Hill, Inc.