

Rotation: Moment of Inertia and Torque

Every time we push a door open or tighten a bolt using a wrench, we apply a force that results in a rotational motion about a fixed axis. Through experience we learn that where the force is applied and how the force is applied is just as important as how much force is applied when we want to make something rotate. This tutorial discusses the dynamics of an object rotating about a fixed axis and introduces the concepts of torque and moment of inertia. These concepts allows us to get a better understanding of why pushing a door towards its hinges is not very a very effective way to make it open, why using a longer wrench makes it easier to loosen a tight bolt, etc.

This module begins by looking at the kinetic energy of rotation and by defining a quantity known as the moment of inertia which is the rotational analog of mass. Then it proceeds to discuss the quantity called torque which is the rotational analog of force and is the physical quantity that is required to changed an object's state of rotational motion.

Moment of Inertia

Kinetic Energy of Rotation

Consider a rigid object rotating about a fixed axis at a certain angular velocity. Since every particle in the object is moving, every particle has kinetic energy. To find the total kinetic energy related to the rotation of the body, the sum of the kinetic energy of every particle due to the rotational motion is taken. The total kinetic energy K can be expressed as

$$K = \frac{1}{2} m_1 \cdot v_1^2 + \frac{1}{2} m_2 \cdot v_2^2 + \frac{1}{2} m_3 \cdot v_3^2 + \dots + \frac{1}{2} m_n \cdot v_n^2 \quad \dots \text{Eq. (1)}$$

where, n is the total number of particles the rigid body has been subdivided into. This equation can be written as

$$K = \sum_{i=1}^n \frac{1}{2} m_i \cdot v_i^2$$

... Eq. (2)

where, m_i is the mass of the i^{th} particle and v_i is the speed of the i^{th} particle. Since $v_i = \omega \cdot r_i$, this equation can be further written as

$$K = \sum_{i=1}^n \frac{1}{2} m_i \cdot \omega^2 \cdot r_i^2$$

... Eq. (3)

or

$$K = \frac{1}{2} \left(\sum_{i=1}^n m_i \cdot r_i^2 \right) \cdot \omega^2$$

... Eq. (4)

Here, r_i is the distance of the i^{th} particle from the axis of rotation. This equation resembles the kinetic energy equation of a rigid body in linear motion, $K = \frac{1}{2} M \cdot v_{com}^2$ and the term in parenthesis is the rotational analog of total mass and is called the moment of inertia.

$$I = \sum_{i=1}^n m_i \cdot r_i^2$$

... Eq. (5)

Eq. (4) can now be further simplified to

$$K = \frac{1}{2} \cdot I \cdot \omega^2$$

... Eq. (6)

As can be see from Eq. (5), the moment of inertia depends on the axis of rotation. It is only constant for a particular rigid body and a particular axis of rotation.

Calculating Moment of Inertia

Integration can be used to calculate the moment of inertia for many different shapes. Eq. (5) can be rewritten in the following form,

$$I = \sum_{i=1}^n m_i \cdot r_i^2 = \int r^2 dm$$

... Eq. (7)

where r is the distance of a differential mass element dm from the axis of rotation.

Example 1: Moment of Inertia of a Disk About its Central Axis

Problem Statement: Find the moment of inertia of a disk of radius R , thickness t , total mass M , and total volume V about its central axis as shown in the image below.

Solution:

The disk can be divided into a very large number of thin rings of thickness t and a differential width dr . The volume of one of these rings, of radius r , can be written as

$$dV = (2 \cdot \pi \cdot r) \cdot t \cdot dr$$

and the mass can be written as

$$dm = \rho \cdot dV$$

where, $\rho = \frac{M}{V}$ is the density of the solid.

Since every particle in the ring is located at the same distance r from the axis of rotation, the moment of inertia of this ring can be written as

$$dI = \rho \cdot r^2 \cdot dV$$

Integrating this for the entire disk, gives

$$\begin{aligned} I_{disk} &= \int_0^R dI = \int_0^R \rho \cdot r^2 \cdot (2 \cdot \pi \cdot r) \cdot t \cdot dr \\ &= \frac{1}{2} \cdot \rho \cdot \pi \cdot t \cdot R^4 = \frac{1}{2} \cdot (\rho \cdot \pi \cdot t \cdot R^2) \cdot R^2 \end{aligned}$$

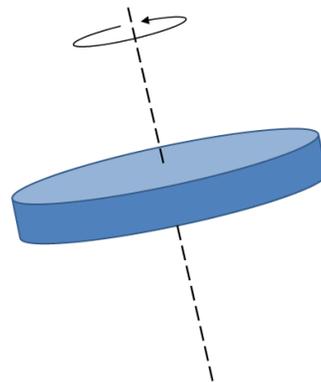


Fig. 1: Disk rotating about central axis.

Since $\pi \cdot t \cdot R^2 = V$ and $\rho \cdot V = M$, the moment of inertia of the disk is

$$I_{disk} = \frac{M}{2} \cdot R^2$$

... Eq. (8)

Parallel Axis Theorem

If the moment of inertia of an object about an axis of rotation that passes through its center of mass (COM) is known, then the moment of inertia of this object about any axis parallel to this axis can be found using the following equation:

$$I = I_{COM} + Mh^2$$

... Eq. (9)

where, h is the distance between the two axes and M is the total mass of the object. This equation is known as the Parallel Axis Theorem.

Proof

Fig. 2 shows an arbitrary object with two coordinate systems. One coordinate system is located on the axis of interest passing through the point P and the other is located on the axis that passes through the center of mass (COM). The coordinates of a differential element with respect to the axis through P is (x,y) and with respect to the axis through the COM is (x',y') . The distance between the two axes is h .

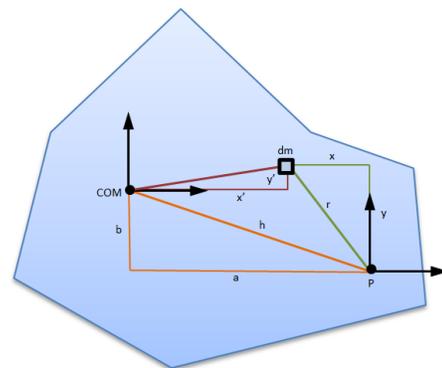


Fig. 2: Parallel axes.

The moment of inertia of the object

about an axis passing through P can be written as

$$I = \int r^2 dm = \int (x^2 + y^2) dm$$

This can be further written as

$$I = \int ((x' - a)^2 + (y' + b)^2) dm$$

Rearranging the terms inside the integral we get

$$I = \int (x'^2 + y'^2) dm + \int (a^2 + b^2) dm - 2 \cdot a \cdot \int x' dm + 2 \cdot b \cdot \int y' dm$$

The last two terms are equal to 0 because, by definition, the COM is the location where $\int x' dm$ and $\int y' dm$ are zero. This equation then simplifies to

$$I = I_{COM} + M \cdot h^2$$

which is the Parallel Axis Theorem.

Example 2: Moment of Inertia of a disk about an axis passing through its circumference

Problem Statement: Find the moment of inertia of a disk rotating about an axis passing through the disk's circumference and parallel to its central axis, as shown below. The radius of the disk is R, and the mass of the disk is M.

Using the parallel axis theorem and the equation for the moment of inertia of a disk about its central axis developed in the previous example, Eq. (8), the moment of inertia of the disk about the specified axis is

$$I = \frac{1}{2} \cdot M \cdot R^2 + M \cdot R^2 = \frac{3}{2} \cdot M \cdot R^2$$

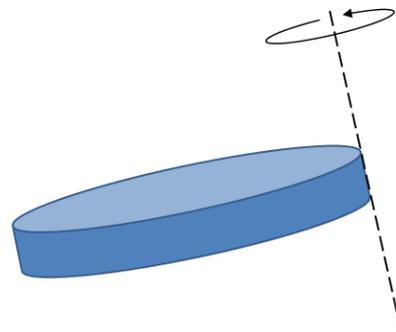


Fig. 3: Disk rotating about an axis passing through the circumference.

Torque and Newton's Second Law for Rotation

Torque, also known as the moment of force, is the rotational analog of force. This word originates from the Latin word *torquere* meaning "to twist". In the same way that a force is necessary to change a particle or object's state of motion, a torque is necessary to change a particle or object's state of rotation. In vector form it is defined as

$$\vec{\tau} = \vec{r} \times \vec{F}$$

... Eq. (10)

where $\vec{\tau}$ is the torque vector, \vec{F} is the force vector and \vec{r} is the position vector of the point where the force is applied relative to the axis of rotation. The direction of the torque is always perpendicular to the plane in which it is applied, hence, for two dimensional rotation this can be simplified to

$$\tau = r \cdot F \cdot \sin(\phi)$$

... Eq. (11)

where r is the distance between the axis of rotation and the point at which the force is applied, F is the magnitude of the force and ϕ is the angle between the position vector of the point at which the force is applied (relative to the axis of rotation) and the direction in which the force is applied. The direction of this torque is perpendicular to the plane of rotation. Eq. (11) shows that the torque is maximum when the force is applied perpendicular to the line joining the point at which the force is applied and the axis of rotation.

Newton's Second Law for Rotation

Analogous to Newton's Second Law for a particle, $\vec{F}_{net} = \frac{d\vec{p}}{dt}$ (more commonly written as $F = ma$ for constant mass), where $\vec{p} = m \cdot \vec{v}$ is the linear momentum, the following

equation is Newton's Second Law for rotation in vector form.

$$\vec{\tau}_{net} = \frac{d\vec{l}}{dt}$$

... Eq. (12)

where

$$\vec{l} = (\vec{r} \times \vec{p})$$

... Eq. (13)

is the quantity analogous to linear momentum known as the angular momentum. If the net torque is zero, then the rate of change of angular momentum is zero and the angular momentum is conserved.

In two dimensions, for a rigid body, this reduces to

$$\tau_{net} = I \cdot \alpha$$

... Eq. (14)

Not only is Eq. (14) analogous to $F = ma$, it is also just a special form of this equation applied to rotation. The following subsection shows a simple derivation of Eq. (10) and Eq. (14).

▼ Brief development of the torque equations.

Consider a particle with a momentum \vec{p} and a position vector of \vec{r} measured from the axis of rotation.

If we define a variable \vec{l} , such that

$$\vec{l} = (\vec{r} \times \vec{p})$$

differentiating both sides gives

$$\frac{d\vec{l}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt} + \frac{d\vec{r}}{dt} \times \vec{p}$$

This can be re-written as

$$\frac{d\vec{l}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt} + \vec{v} \times (m \cdot \vec{v})$$

and, since $\vec{F} = \frac{d\vec{p}}{dt}$ and the cross product of a vector with itself is 0, this equation reduces to

$$\frac{d\vec{l}}{dt} = \vec{r} \times \vec{F}$$

Now, if we define $\frac{d\vec{l}}{dt}$ as something called torque, represented by $\vec{\tau}$, we get

$$\vec{\tau} = \vec{r} \times \vec{F}$$

This is Eq. (10). Continuing with this equation,

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times \frac{d\vec{p}}{dt}$$

we can write

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times \left(m \cdot \frac{d\vec{v}}{dt} \right)$$

Since $\frac{d\vec{v}}{dt} = \vec{a} = \vec{\alpha} \times \vec{r}$,

$$\vec{\tau} = \vec{r} \times \vec{F} = m \cdot [\vec{r} \times (\vec{\alpha} \times \vec{r})]$$

Using an identity for cross products,

$$\vec{\tau} = \vec{r} \times \vec{F} = m \cdot [\vec{\alpha} (\vec{r} \cdot \vec{r}) - \vec{r} (\vec{r} \cdot \vec{\alpha})]$$

this simplifies to

$$\vec{\tau} = \vec{r} \times \vec{F} = m \cdot r^2 \cdot \vec{\alpha}$$

and, finally, we get

$$\vec{\tau} = \vec{r} \times \vec{F} = I \cdot \vec{\alpha}$$

Although this is only a proof for a single particle, a similar method will give the same result for larger rigid bodies composed of a large number of particles.

The beauty of all these equations is that, even for large complex geometries (not considering relativistic effects), they are all based on Newton's three fundamental laws of motion.

Returning to the topic of doors and wrenches, why is pushing a door towards its hinges is not very a very effective way to make it open? This questions can be answered using Eq. (11). If a door is pushed, as shown in Fig. 4, then the torque is maximum when ϕ is 90° and decreases as ϕ changes. If the door is pushed towards its hinges, then ϕ is 180° which makes the torque equal to 0.

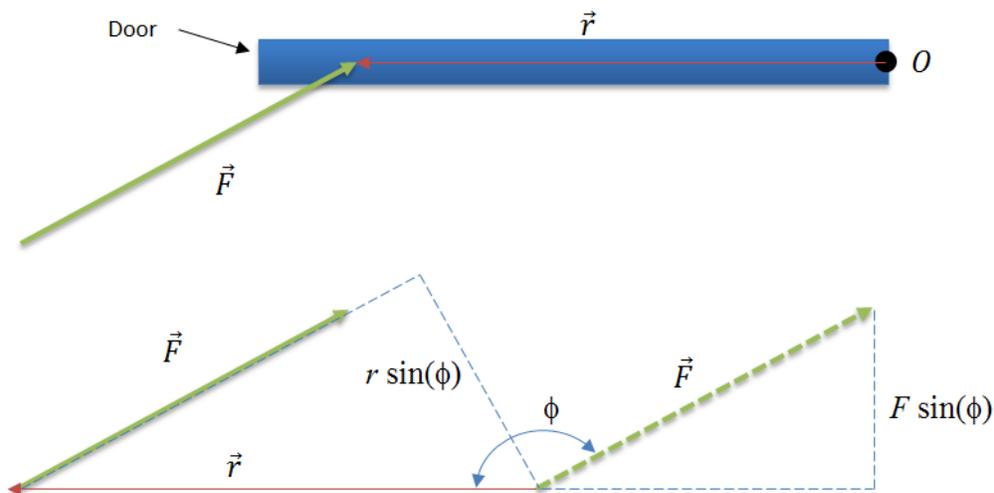


Fig. 4: Torque on a door

Similarly, using Eq. (11), a longer wrench makes it easier to loosen a tight bolt because increasing r allows for a greater torque.

Examples with MapleSim

Example 3: Stationary Bicycle

Problem statement: The flywheel of a stationary exercise bicycle is made of a solid iron disk of radius 0.2m and thickness 0.02m. A person applies a torque that has an initial value of 25 Nm and decreases at the rate of 5 Nm/s for a total time of 5 seconds.

- What is the moment of inertia of the wheel?
- What is the initial angular acceleration of the wheel?
- What is the rate (in rpm) at which the person needs to pedal after 3 seconds to be able to continue applying the torque?
- How much energy has the person converted to the rotational kinetic energy of the wheel after 5 seconds?

Analytical Solution

restart :

Data:

$$R := 0.2 : \quad [\text{m}]$$

$$T := 0.02 : \quad [\text{m}]$$

$$\rho := 7700 : \quad [\text{kg/m}^3]$$

$$\tau := 25 - 5 \cdot t : \quad [\text{Nm}]$$

$$t_1 := 3 : \quad [\text{sec}]$$

$$t_2 := 5 : \quad [\text{sec}]$$

Solution:

Part a) Calculating the moment of inertia of the wheel.

Using Eq. (8), derived in the moment of inertia example, the moment of inertia of the disk is

$$I_{disk} := \frac{(\text{Pi} \cdot R^2 \cdot T \cdot \rho)}{2} \cdot R^2 = 3.94 \pi \xrightarrow{\text{at 5 digits}} 12.38$$

Therefore, the moment of inertia of the disk is 12.38 kgm².

Part b) Determining the initial angular acceleration of the wheel.

The initial angular acceleration can be found using Eq. (14).

$$\alpha_0 := \frac{50}{I_{disk}} = \frac{12.68}{\pi} \xrightarrow{\text{at 5 digits}} 4.04$$

Therefore, the initial angular acceleration of the wheel is 4.04 rad/s².

Part c) Determining the angular velocity after 3 seconds.

To be able to continue applying the torque, the person must be able to match the angular velocity of the wheel. The equation for angular velocity can be obtained by integrating the equation for angular acceleration,

$$\alpha := \frac{\tau}{I_{disk}} :$$

$$\omega := \int_0^t \alpha dt$$

$$- .20 t^{2.00} + 4.04 t \quad (3.1.1.3.1)$$

$$\omega_3 := eval(\omega, t=3)$$

$$10.29 \quad (3.1.1.3.2)$$

Converting this value from rad/s to rpm,

$$n_3 := \frac{\omega_3 \cdot 60}{\text{Pi}}$$

$$\frac{617.66}{\pi} \quad (3.1.1.3.3)$$

$\xrightarrow{\text{at 5 digits}}$

$$196.61 \quad (3.1.1.3.4)$$

Therefore, the person needs to pedal at 196.61 rpm at 3 seconds.

Part d) Determining the rotational kinetic energy after 5 seconds

Eq. (6) can be used to find the rotational kinetic energy of the wheel. To find the angular velocity of the wheel after 5 seconds, the same approach in Part c) can be used.

$$\omega_5 := \text{eval}(\omega, t = 5)$$

15.14 (3.1.1.4.1)

Using Eq. (6),

$$K_5 : \frac{1}{2} \cdot I_{disk} \cdot \omega_5^2$$

451.76 π (3.1.1.4.2)

at 10 digits \rightarrow

1419.26 (3.1.1.4.3)

Therefore, after 5 seconds, the person has input 1.4 kJ of energy into the wheel.

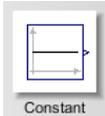
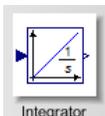
MapleSim Simulation

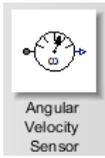
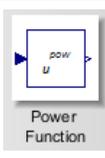
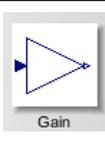
Constructing the Model

Step1: Insert Components

Drag the following components into the workspace:

Table 1: Components and locations.

Component	Location
 Constant	Signal Blocks > Common
 Integrator	Signal Blocks > Common
 Torque	1-D Mechanical > Rotational

	> Torque Drivers
 Angular Velocity Sensor	1-D Mechanical > Rotational > Sensors
 Power Function	Signal Blocks > Mathematical > Functions
 Gain	Signal Blocks > Common

Step 2: Connect the components

Connect the components as shown in the following diagram (the dashed boxes are not part of the model, they have been drawn on top to help make it clear what the different components are for).

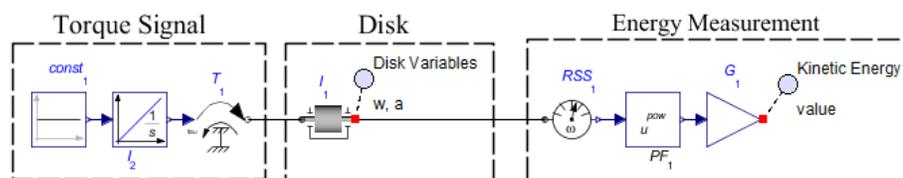


Fig. 5: Component diagram for the Stationary Bicycle example

Step 3: Set the Parameters

1. Click the **Constant** component and (in the **Inspector** tab) set the parameter k to **-5**. This is the rate at which the torque is decreasing.
2. Click the **Integrator** component and enter **50** for the initial value (y_0). This is the initial value of the torque. This component integrates -5 to give $-5t$ and adds 50 to it, giving $f(t)=50 - 5t$ which the applied torque.

3. Click the **Inertia** component and enter **12.38 kg·m²** for the moment of inertia (J).
4. Click the **Gain** component and set the parameter k to **1/2*12.38**.
5. The parameter (*power*) of the **Power Function** component should be left to the default value of 2.

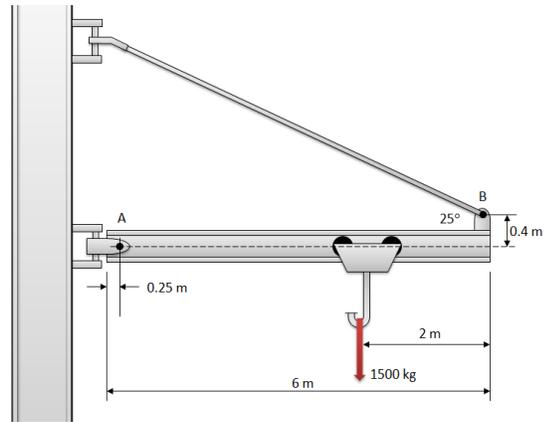
Note: If desired, these parameters can be set as variables and a parameter block can be created by clicking the **Add a Parameter Block** icon () and then clicking on the workspace. The values for the parameters can then be entered using this block, which is useful if you want to repeatedly change the parameters and see the effect on the simulation results.

Step 4: Run the Simulation

1. Click the **Probe** attached to the **Inertia** component and select **Speed** and **Acceleration** in the **Inspector** tab.
2. In the **Settings** tab reduce the **Simulation duration (t_d)** to **5 s**.
3. Click **Run Simulation** () .
4. The plots generated can be used to find the solutions to the problem.

Example 4: Jib Crane

Problem Statement: A jib crane, as shown in Fig. 6, consists of a beam supported at A by a pin joint and at B by a cable. There is an object hanging from a hook as shown in the image. The total weight of the object and the hook assembly is 1500 kg and the total weight of the



beam is 500 kg. Find the tension in the cable.

Fig. 6: Jib crane

Analytical Solution

restart;

Data:

$$M_{object} := 1500 : \quad [\text{kg}]$$

$$M_{beam} := 500 : \quad [\text{kg}]$$

$$g := 9.81 : \quad [\text{m/s}^2]$$

$$L_1 := 0.25 : \quad [\text{m}]$$

$$L_2 := 6 : \quad [\text{m}]$$

$$L_3 := 2 : \quad [\text{m}]$$

$$L_4 := 0.4 : \quad [\text{m}]$$

$$\theta := \frac{25}{180} \cdot \text{Pi} : \quad (\text{Converting the angle from degrees to radians so that it can be used with the inbuilt trigonometric functions. } 1 \text{ deg} = \text{Pi}/180 \text{ rad.})$$

Solution:

The torque due to the tension in the cable can be expressed using Eq. (11). The angle between the line that joins AB and the force vector, as shown in Fig.7, is

$$\phi_{cable} := \theta + \tan^{-1}\left(\frac{0.4}{5.75}\right) = \frac{5}{36} \pi + 0.06945332599 \xrightarrow{\text{at 5 digits}} .506$$

Therefore,

$$\tau_{cable} := \sqrt{0.4^2 + (6 - 0.25)^2} \cdot T \cdot \sin(\phi_{cable}) = 5.76 T \sin(.14 \pi + .07)$$

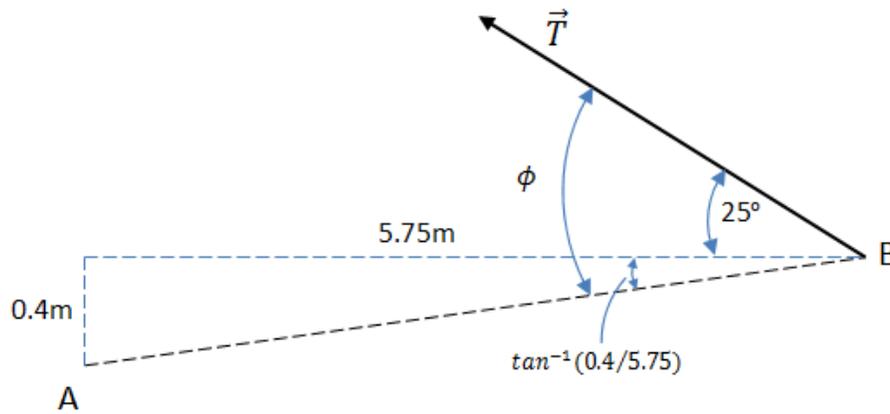


Fig. 7: Angle for the Tension vector

Assuming a uniform beam, the force due to the beam's weight will act on the center of mass. Hence the torque due to the weight of the beam is

$$\tau_{beam} := \left(\frac{6}{2} - 0.25\right) \cdot M_{beam} \cdot g \cdot \sin\left(\frac{\text{Pi}}{2}\right) = 13488.75$$

Also, the torque due to the object and hook assembly is

$$\tau_{object} := (6 - 0.25 - 2) \cdot M_{object} \cdot g \cdot \sin\left(\frac{\text{Pi}}{2}\right) = 55181.25$$

Since the beam is stationary and not accelerating, it is in a state of static equilibrium. From Eq. (14), this means that the net torque on the beam is equal to zero. Hence, the sum of the torque on the beam, about point A, from the tension in the cable, from

the weight on the hook and from the weight of the beam is zero.

$$\tau_{cable} - \tau_{beam} - \tau_{object} = 0$$

$$5.763896252 T \sin\left(\frac{5}{36} \pi + 0.06945332599\right) - 68670.0000 = 0 \quad (3.2.1.1)$$

→ solve for T

$$[[T = 24.59 \times 10^3]] \quad (3.2.1.2)$$

Therefore, the tension in the cable is 24.59 kN.

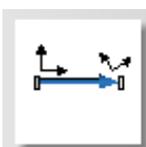
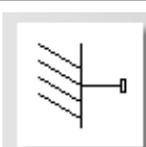
MapleSim Simulation

Constructing the Model

Step1: Insert Components

Drag the following components into the workspace:

Table 2: Components and locations

Component	Location
 Revolute (3 required)	Multibody > Joints and Motions
 Rigid Body Frame (6 required)	Multibody > Bodies and Frames
 Fixed Frame (2 required)	Multibody > Bodies and Frames

Step 2: Connect the components

Connect the components as shown in the following diagram.

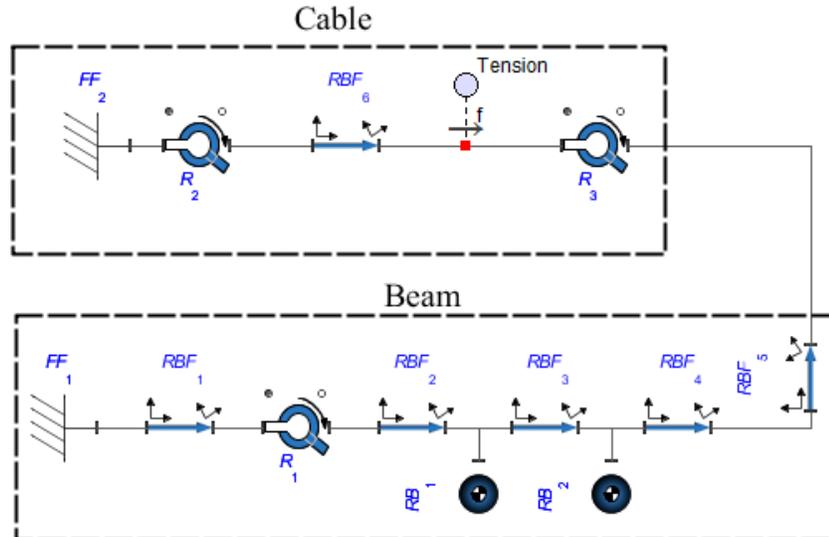


Fig. 8: Component diagram for the jib crane example

Step 3: Set up the beam

1. Change the x,y,z offsets (click the component and enter the values for \vec{r}_{XYZ} in the **Inspector** tab) of **RBF₁**, **RBF₂**, **RBF₃**, **RBF₄**, and **RBF₅** to **[0.25,0,0]**, **[2.75, 0.0]**, **[1,0,0]**, **[2,0,0]** and **[0,0.4,0]** respectively.
2. Enter **500 kg** for the mass (m) of **RB₁** and **1500 kg** for the mass (m) of **RB₂**.
Here, **RB₁** corresponds to the weight of the beam and **RB₂** corresponds to the weight of the object and the hook assembly.
3. The x,y,z offset (\vec{r}_{XYZ}) of **FF₁** and the axis of rotation (\hat{e}_1) of **R₁** should be left to the default values which are **[0,0,0]** and **[0,0,1]** respectively.

Step 4: Set up the cable

1. Enter **[0,0.4+6*tan(25/180*Pi),0]** for the x,y,z offset (\vec{r}_{XYZ}) of **FF₂**.
2. Enter **[6/cos(25/180*Pi),0,0]** for the x,y,z offset (\vec{r}_{XYZ}) of **RBF₆**.

3. The axes of rotation (\hat{e}_1) of R_2 and R_3 can be left to the default values.
4. Click the **Probe** placed between RBF_6 and R_3 , and select **1** under **Force**. This will show the force along the direction of the rigid body frame that represents the cable.

Step 5: Run the simulation

The following image shows the 3-D view of the model.

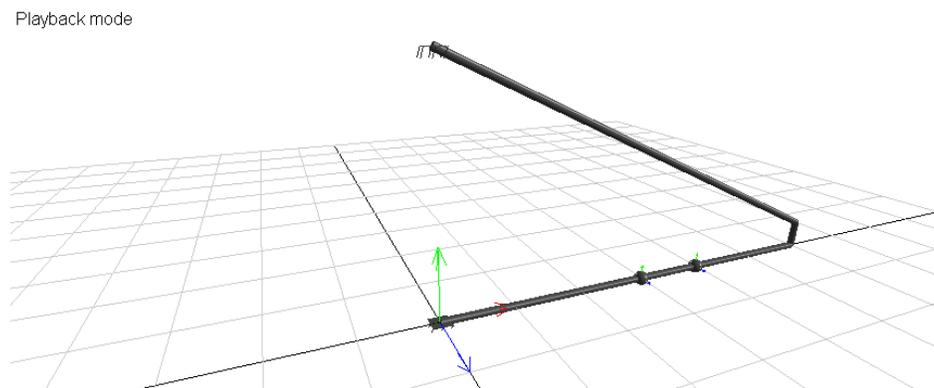


Fig. 9: 3D view of the Jib crane model

Gyroscopic Precession Simulation

This simulation shows an interesting phenomena that illustrates how torque and angular momentum behave as vectors. A gyroscope is basically a spinning wheel or disk that, due to its angular momentum, behaves very differently than a non-spinning wheel. Gyroscopes are commonly used in aircrafts and electronics like cell phones to measure rotation and orientation. They are also used to achieve stability in specialized vehicles, camera supports, etc.

For example, if a wheel, that is not spinning, is suspended from one end of its axle by a joint

that is free to rotate in all planes, as shown in Fig. 10, then the wheel will fall and oscillate like a pendulum. However, if the wheel is spinning sufficiently fast, it will rotate in a horizontal plane and will not swing like a pendulum. This motion is called precession. In Fig. 10, the red line (which lies on the horizontal plane) shows the path of the free end of the axle of the spinning wheel, and the blue line (which lies on the vertical plane) shows the path of the free end of the axle of the non-spinning wheel. From a mathematical point of view, this phenomena occurs because of the vector addition of the existing angular momentum of the spinning wheel and the angular momentum that is added due to the torque (due to gravity). The torque on the wheel, in the orientation shown below, points in a direction perpendicular to the existing angular momentum of the spinning wheel (right-hand-rule). Hence it increases the angular momentum in a direction perpendicular to it. This changes the angular momentum and causes the wheel to rotate in the horizontal plane instead of falling and oscillating in the vertical plane.

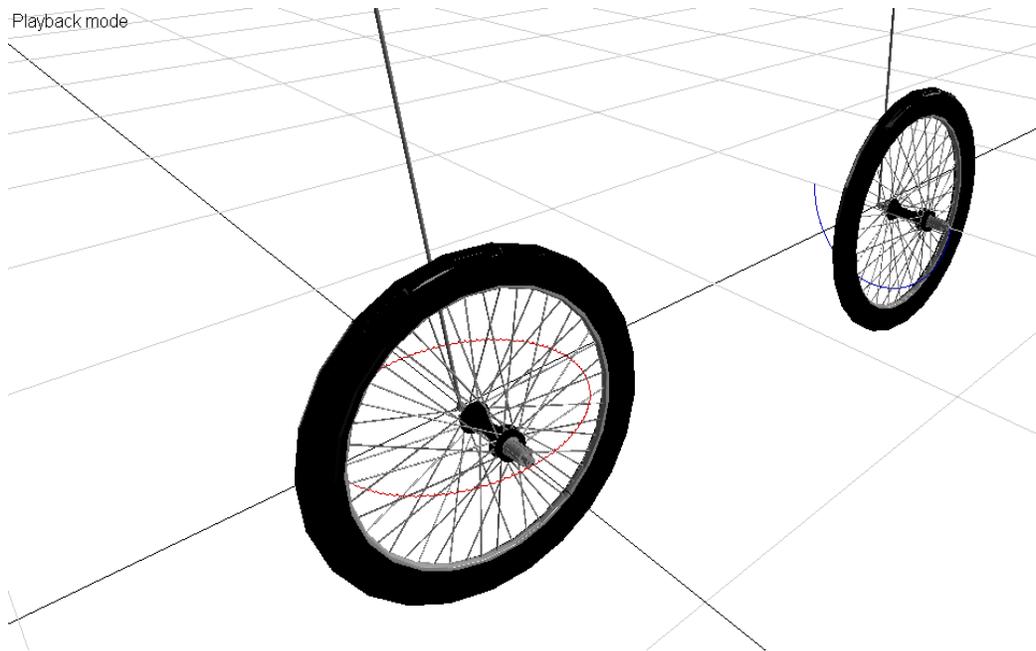
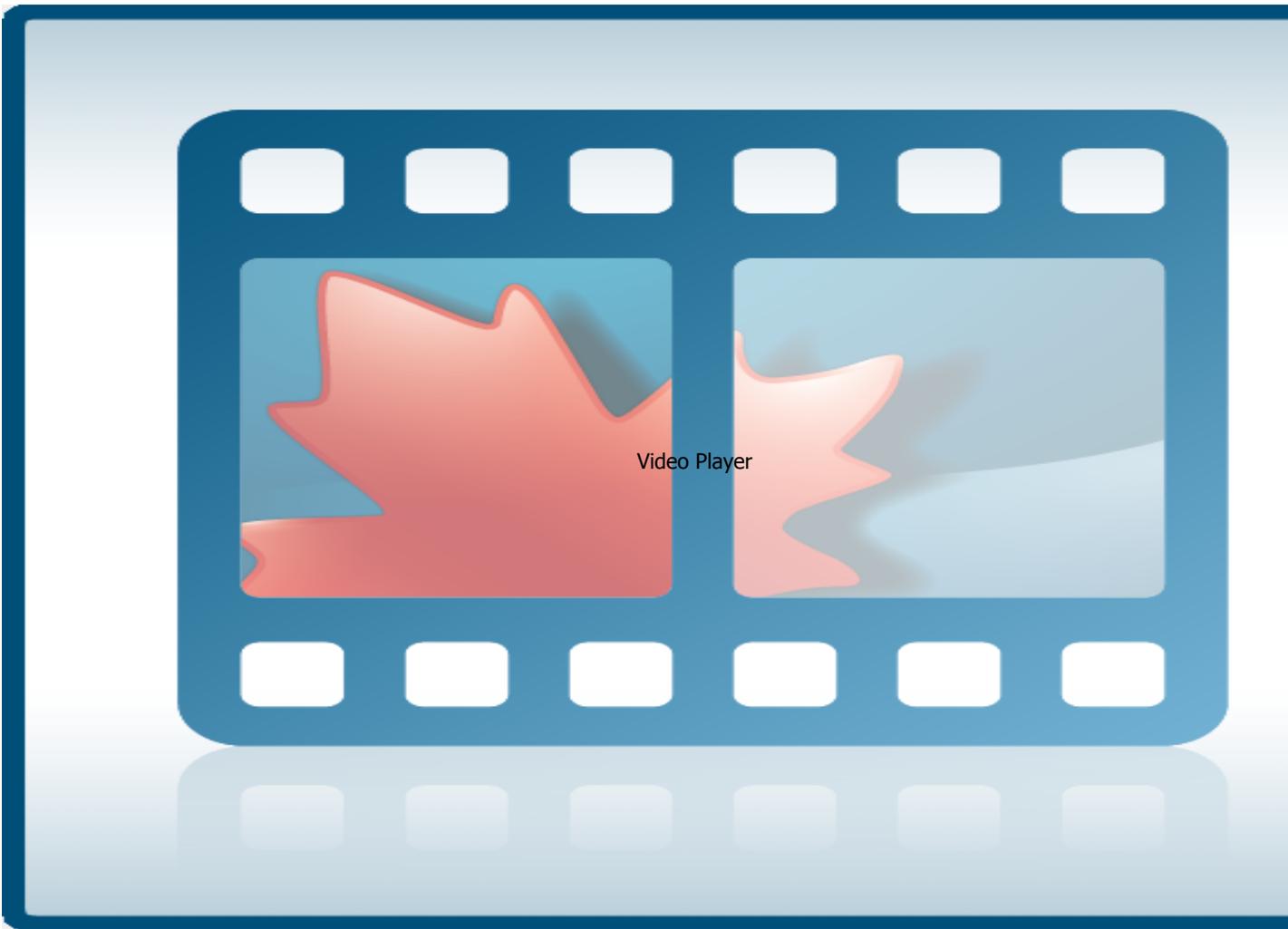


Fig. 10: 3D view of the gyroscope simulation.

The following figure shows the component diagram of the model of the spinning wheel.



Video 1: 3-D visualization of a spinning wheel (showing gyroscopic precession) and a non-spinning wheel.

References:

1. Halliday et al. "Fundamentals of Physics", 7th Edition. 111 River Street, NJ, 2005, John Wiley & Sons, Inc.
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