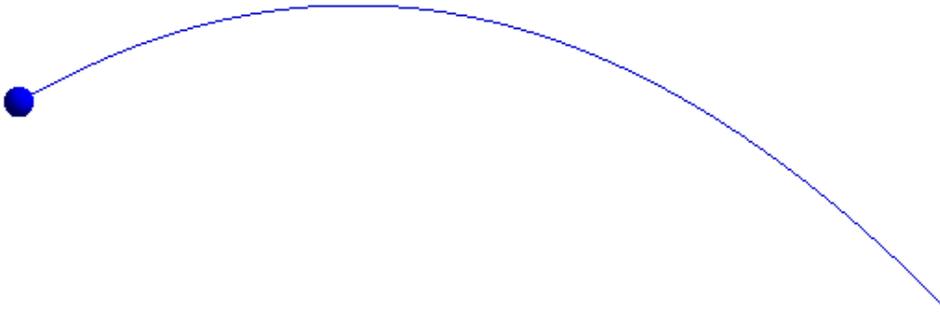


Projectile Motion



Projectile motion is a special case of two-dimensional motion. A particle moving in a vertical plane with an initial velocity and experiencing a free-fall (downward) acceleration, displays projectile motion. Some examples of projectile motion are the motion of a ball after being hit/thrown, the motion of a bullet after being fired and the motion of a person jumping off a diving board. For now, we will assume that the air, or any other fluid through which the object is moving, does not have any effect on the motion. In reality, depending on the object, air can play a very significant role. For example, by taking advantage of air resistance, a parachute can allow a person to land safely after jumping off an airplane. These effects are very briefly discussed at the end of this module.

▼ Projectile Motion Analysis

Before proceeding, the following subsection provides a reminder of the three main equations of motion for constant acceleration. These equations are used to develop the equations for projectile motion.

▼ Equations of Motion for Constant Acceleration

The following equations are three commonly used equations of motion for an object moving with a constant acceleration.

$$v = v_0 + a \cdot t$$

... Eq. (1)

$$x - x_0 = v_0 \cdot t + \frac{1}{2} \cdot a \cdot t^2$$

... Eq. (2)

$$v^2 = v_0^2 + 2 \cdot a \cdot (x - x_0)$$

... Eq. (3)

Here, a is the acceleration, v is the speed, v_0 is the initial speed, x is the position, x_0 is the initial position and t is the time.

To begin, consider an object with an initial velocity $\vec{v}_0 = v_{0x} \hat{i} + v_{0y} \hat{j}$ launched at an angle of θ_0 measured from the positive x-direction. The x- and y-components of the object's initial velocity, v_{0x} and v_{0y} , can be written as

$$v_{0x} = v_0 \cdot \cos(\theta_0) \quad \text{and} \quad v_{0y} = v_0 \cdot \sin(\theta_0)$$

... Eq. (4) and Eq. (5)

Here, the x-axis corresponds to the horizontal direction and the y-axis corresponds to the vertical direction. Since there is a downward acceleration due to gravity, the y-component of the object's velocity is continuously changing. However, since there is no horizontal acceleration, the x-component of the object's velocity stays constant throughout the motion. Breaking up the motion into x- and y-components simplifies the analysis.

The Horizontal Component

Horizontal Displacement

Using Eq. (2), at a time t , the horizontal displacement for a projectile can be written as

$$x - x_0 = v_{0x} \cdot t + \frac{1}{2} \cdot a_x \cdot t^2$$

... Eq. (6)

where, x_0 is the initial position and a_x is the horizontal acceleration. Since $a_x = 0$ and $v_{0x} = v_0 \cdot \cos(\theta_0)$, the above equation reduces to

$$x - x_0 = v_0 \cdot \cos(\theta_0) \cdot t$$

... Eq. (7)

Eq. (7) is the equation for the horizontal displacement of a projectile as a function of time. This shows that, before the projectile hits the ground or encounters any other resistance, the horizontal displacement changes linearly with time.

Horizontal Velocity

Using Eq. (1), the horizontal velocity can be written as

$$v_x = v_{0x} + a_x \cdot t$$

... Eq. (8)

which further reduces to

$$v_x = v_0 \cdot \cos(\theta_0)$$

... Eq. (9)

Since v_0 and θ_0 are constants, this equation shows that the horizontal velocity remains unchanged throughout the motion.

The Vertical Component

Vertical Displacement

Using Eq. (2), at a time t , the vertical displacement for the projectile can be written as

$$y - y_0 = v_{0y} \cdot t + \frac{1}{2} \cdot a_y \cdot t^2$$

... Eq. (10)

where, y_0 is the initial position and a_y is the vertical acceleration. Since $a_y = -g$ and $v_{0y} = v_0 \cdot \sin(\theta_0)$, the above equation reduces to

$$y - y_0 = v_0 \cdot \sin(\theta_0) \cdot t - \frac{1}{2} \cdot g \cdot t^2$$

... Eq. (11)

Eq. (11) is the equation for the vertical displacement of a projectile as a function of time. Unlike the horizontal displacement, the vertical displacement does not vary linearly with time.

Vertical Velocity

Using Eq. (1), the vertical velocity can be written as

$$v_y = v_{0y} + a_y \cdot t$$

... Eq. (12)

Since $a_y = -g$ and $v_{0y} = v_0 \cdot \sin(\theta_0)$, this can be further written as

$$v_y = v_0 \cdot \sin(\theta_0) - g \cdot t$$

... Eq. (13)

This equation shows that the vertical component of the velocity continuously changes with time.

Also, using Eq. (3), the vertical velocity can be expressed as

$$v_y^2 = v_{0y}^2 + 2 \cdot a_y \cdot (y - y_0)$$

... Eq. (14)

which can be further written as

$$v_y^2 = (v_0 \cdot \sin(\theta_0))^2 - 2 \cdot g \cdot (y - y_0)$$

... Eq. (15)

This equation provides a relation between the vertical velocity and the vertical displacement.

Maximum Vertical Displacement

When an object is launched with a positive vertical velocity component, the vertical velocity becomes zero at the point where the maximum height is reached. Hence, substituting $v_y = 0$ in Eq. (15) gives the following equation for the maximum vertical displacement of a projectile.

$$(y - y_0) = \frac{(v_0 \cdot \sin(\theta_0))^2}{2 \cdot g}$$

... Eq. (16)

Also, from Eq. (16) observe that, to achieve the maximum possible vertical displacement for a fixed initial velocity, $(\sin(\theta_0))^2$ should be equal to 1. This happens when θ_0 is 90° (-90° is also valid mathematically but not physically. Mathematically, with a launch angle of -90° , the velocity is zero at a negative time). Therefore, to achieve the maximum height, an object must be launched straight up, which is also what intuition would tell us.

The Path Equation

By combining Eq. (7) and Eq. (11), an equation for the projectile's trajectory can be obtained. Rearranging Eq. (7) to isolate for t and substituting it into Eq. (11) gives

$$y = y_0 + (x - x_0) \cdot \tan \theta_0 - \frac{g \cdot (x - x_0)^2}{2(v_0 \cdot \cos(\theta_0))^2}$$

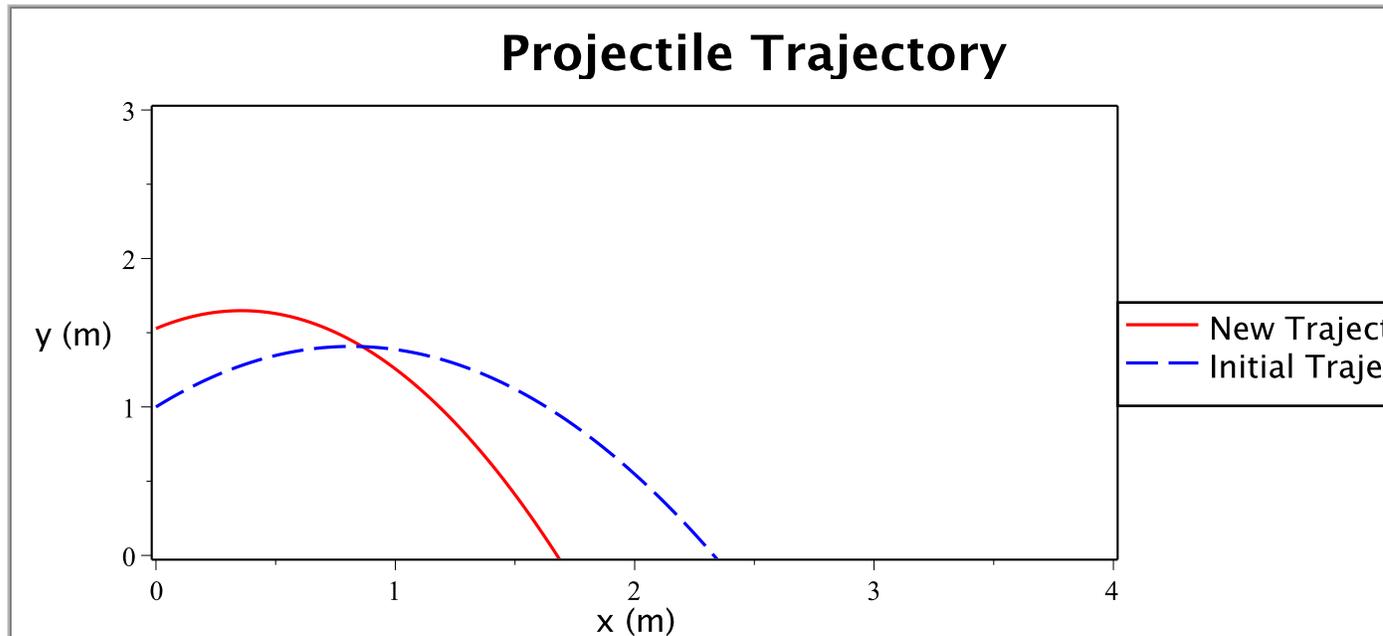
... Eq. (17)

This is the equation for the path of a projectile. If the initial conditions are known, then the path of the projectile can be determined. Due to the quadratic form of the equation, each y-position has two corresponding x-positions. These different x-positions correspond to different time values. For example, when a ball is thrown, it can cross a certain height

twice, once on the way up and once on the way down. It should also be noted that, due to the tan and cos terms, this path equation is not defined at 90° and -90° . This can also be understood by the fact that there will be many y-positions for the same initial x-position.

In the following plot, the dashed line shows the trajectory of a projectile launched at an initial height of 1m, with an initial velocity of 4m/s and at an angle of 45° from the horizontal. The sliders can be used to adjust the initial conditions and observe the new trajectory (solid line).

Initial height (m)	Launch angle (deg)	Initial velocity (m/s)
 0.0 0.5 1.0 1.5 2.0	 -90.0 45.0 90.0	 0.0 1.0 2.0 3.0 4.0



Horizontal Range

The horizontal range can be defined as the horizontal distance a projectile travels before returning to its launch height. When the final height is equal to the launch height (i.e. $y - y_0 = 0$), Eq. (11) reduces to the following,

$$0 = v_0 \cdot \sin(\theta_0) \cdot t - \frac{1}{2} \cdot g \cdot t^2$$

... Eq. (18)

When t is non-zero, this equation can be written as

$$t = \frac{2 \cdot v_0 \cdot \sin(\theta_0)}{g}$$

... Eq. (19)

By substituting Eq. (18) into Eq. (7), the following equation is obtained

$$x - x_0 = \frac{2 \cdot v_0^2}{g} \cdot \sin(\theta_0) \cdot \cos(\theta_0)$$

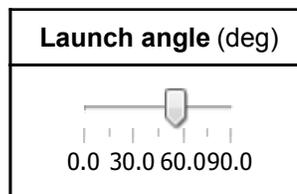
... Eq. (20)

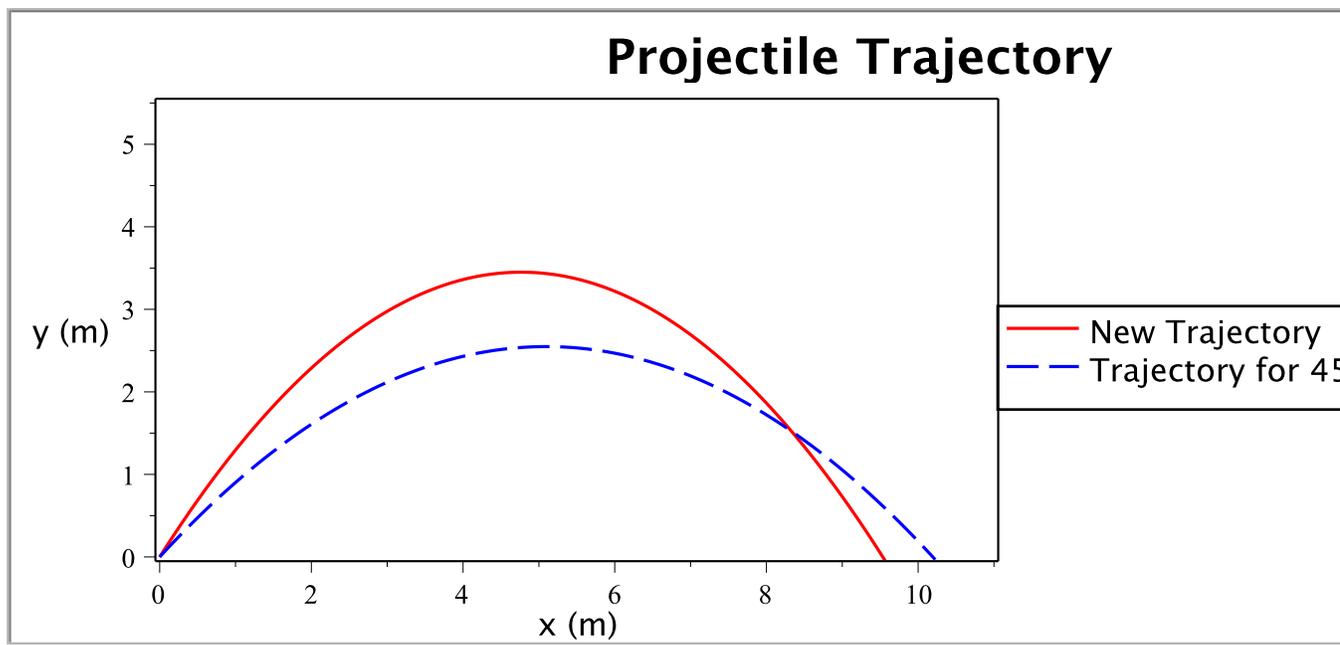
Using the identity $\sin(2 \cdot \theta_0) = 2 \cdot \sin(\theta_0) \cdot \cos(\theta_0)$ gives

$$x - x_0 = \frac{2 \cdot v_0^2}{g} \cdot \sin(2 \cdot \theta_0)$$

... Eq. (21)

This shows that the horizontal displacement, when the projectile returns to the launch height, is greatest when $\sin(2 \cdot \theta_0) = 1$. This means that the range is maximum when the launch angle is 45° . The following plot shows the trajectory of a projectile launched with an initial velocity of 10 m/s, at an angle of 45° and with no initial height (dashed line). The launch angle can be varied to observe the effect on the range.





Examples with MapleSim

Example 1: Tennis Serve

Problem statement: A tennis player serves a ball with a speed of 30m/s. The ball leaves the racquet at a height of 2.5m and a horizontal distance of 12.25m from the net. The height of the net is 0.91m.

- If the ball leaves the racquet horizontally, will the ball clear the net?
- If the ball leaves the racquet at an angle of 10° below the horizontal, will the ball clear the net?
- What is the minimum angle at which the ball must leave the racquet for it to clear the net?

Analytical Solution

restart;

Data:

$$v_0 := 30 : \quad [\text{m/s}]$$

$$h_0 := 2.5 : \quad [\text{m}]$$

$$d := 12.25 : \quad [\text{m}]$$

$$h_n := 0.91 : \quad [\text{m}]$$

$$\theta_1 := 0 : \quad [\text{rad}]$$

$$\theta_2 := \frac{-10 \cdot \text{Pi}}{180} : \quad (\text{Converting the angle from degrees to radians so that it can be used in the})$$

$$g := 9.81 : \quad [\text{m/s}^2]$$

Solution:

Part a) Determining if the ball clears the net if it leaves the racquet horizontally.

The time it takes the ball to reach the net can be solved for using Eq. (7).

$$t_1 := \frac{d}{v_0 \cdot \cos(\theta_1)} = .4083$$

The height of the center of the ball when it reaches the plane of the net can be calculated using Eq. (11).

$$y_1 := h_0 + v_0 \cdot \sin(\theta_1) \cdot t_1 - \frac{1 \cdot g \cdot (t_1)^2}{2} = 1.682$$

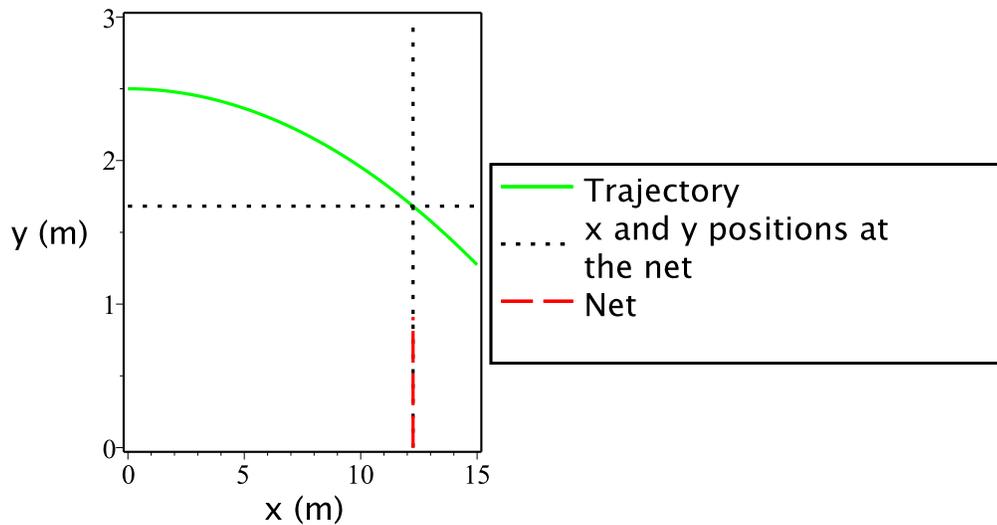
Since y_1 is greater than 0.91m, the ball clears the net for this serve.

In the following plot, the solid line shows the trajectory of the ball for this case. The dotted lines show the x- and y-positions of the ball when it reaches the plane of the net and the dashed line represents the net.



Trajectory - Part a).

Trajectory for Part a.



Part b) Determining if the ball clears the net if it leaves the racquet at an angle 10° below the horizontal.

Using the same approach as Part a), the time that the ball takes to reach the net can be solved for using Eq. (7).

$$t_2 := \frac{d}{v_0 \cdot \cos(\theta_2)} = \frac{0.4083333333}{\cos\left(\frac{1}{18} \pi\right)} \xrightarrow{\text{at 5 digits}} 0.41462$$

The height of the center of the ball when it reaches the plane of the net can be calculated using Eq. (11).

$$y_0 - h_0 = v_0 \cdot \sin(\theta_2) \cdot t_2 - \frac{1 \cdot g \cdot (t_2)^2}{2} :$$

$$y_0 := \text{solve}\left(y_0 - h_0 = v_0 \cdot \sin(\theta_2) \cdot t_2 - \frac{1 \cdot g \cdot (t_2)^2}{2}, y_0\right) = -.5033$$

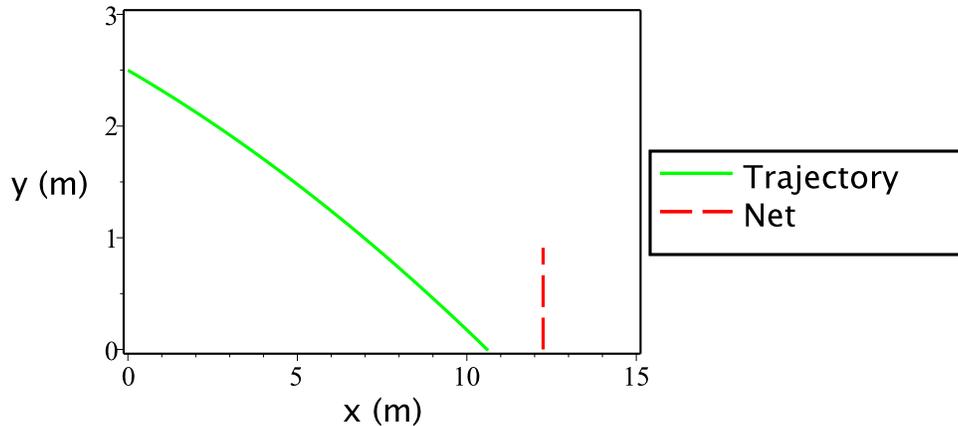
Since y_2 is negative, it means that the ball bounced on the ground before reaching

the net. A negative value is obtained since the equations do not account for the presence of the ground. In the following plot, the solid line shows the trajectory of the ball for this case and the dashed line represents the net.



Trajectory - Part b).

Trajectory for Part b.



Part c) Finding the minimum angle required to clear the net.

For the ball to hit the top of the net, the height of the ball should be equal to the height of the net when the ball reaches the plane of the net. By combining Eq. (7) and Eq. (11), and substituting the parameters specific to this problem, the following equation is obtained.

$$h_n = h_0 + (d) \cdot \tan(\theta_0) - \frac{g \cdot (d)^2}{2(v_0 \cdot \cos(\theta_0))^2} :$$

$$\theta_0 := \text{solve} \left(h_n = h_0 + (d) \cdot \tan(\theta_0) - \frac{g \cdot (d)^2}{2(v_0 \cdot \cos(\theta_0))^2}, \theta_0 \right)$$

$$1.504, -0.063, -1.637, 3.079$$

(2.1.13.1)

Due to the form of the equation, there are many angles that satisfy it. For this case, only the smallest angle in the first and fourth quadrant are of interest. The two angles in the first and fourth quadrant are 1.504 rad and -0.063 rad. The smaller of

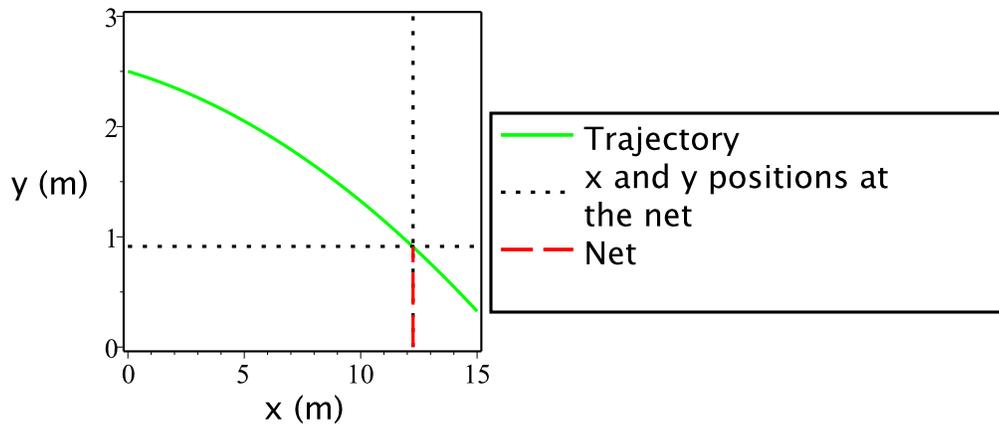
the two angles is -0.063 rad, which is approximately equal to -3.6° . Therefore, the ball must leave the racquet at an angle slightly greater than -3.6° for it to just clear the net.

In the following plot, the solid line shows the trajectory of the ball of this case. The dotted lines show the x- and y- positions of the ball when it reaches the plane of the net and the dashed line represents the net.



Trajectory - Part c).

Trajectory for Part c.



MapleSim Simulation

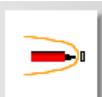
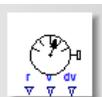
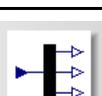
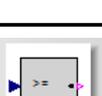
Constructing the model

Step1: Insert Components

Drag the following components into the workspace:

Table 1: Components and locations

Component	Location
	Multibody > Body and Frames

 Rigid Body	
 Path Trace	Multibody > Visualization
 Absolute Translation	Multibody > Sensors
 Real DeMultiplexer	Signal Blocks > Routing > DeMultiplexers
 Greater Equal Threshold	Signal Blocks > Relational
 Terminate	Signal Blocks > Terminate
 Fixed Frame	Multibody > Body and Frames
 Rigid Body Frame	Multibody > Body and Frames

Step 2: Connect the components

Connect the components as shown in the following diagram (the dashed boxes are not part of the model, they have been drawn on top to help make it clear what the

different components are for).

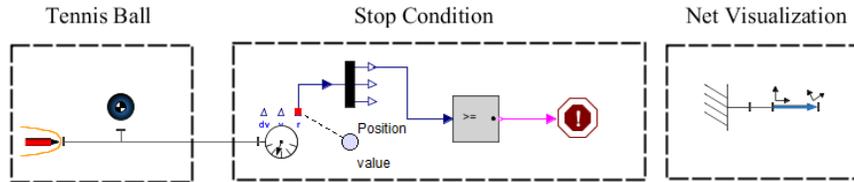


Fig. 1: Component diagram

Step 3: Create parameters

Click the **Add a Parameter Block** icon (), click on the workspace and double click the **Parameters** icon that will appear on the workspace. Create parameters for the initial height of the ball h , the initial speed of the ball v , the distance to the net d , and the launch angle $theta$ (see Fig. 2).

Parameters subsystem default settings

Name	Type	Default Value	Default Units	Description
h	Position [m]	1	m	Initial height (m)
v	Velocity [m/s]	1	m/s	Initial speed (m/s)
d	Position [m]	1	m	Horizontal distance to the net (m)
$theta$	Angle [rad]	1	deg	Launch angle measured from the horizontal (deg)

Fig. 2: Parameter Block settings

Step 4: Adjust the parameters

Return to the main diagram ( > **Main** ) and, with a single click on the **Parameters** icon, enter the following parameters (Fig. 3) in the **Inspector** tab.

Name	<i>Parameters</i>		
Type	Standalone Subsystem		
▼ Parameters			
<i>h</i>	2.5	<i>m</i>	▼
<i>v</i>	30	$\frac{m}{s}$	▼
<i>d</i>	12.25	<i>m</i>	▼
<i>theta</i>	0	<i>deg</i>	▼

Fig. 3: Parameters

Note: Step 3 and Step 4 are not essential and can be skipped. The parameter values can be directly entered for each component instead of using variables. However, creating a parameter block as described above makes it easy to repeatedly change the parameters and play around with the model to see the effects on the simulation results.

Step 5: Change the initial conditions

Return to the main diagram and click the **Rigid Body** component. As shown below, enter the initial conditions in the **Inspector** tab in terms of the parameters.

Name

Type Rigid Body

▼ Parameters

m

$[I]$

$IC_{r,v}$

\vec{r}_0

$Type_v$

\vec{v}_0

$IC_{e,\omega}$

$Type_e$

$\vec{\theta}_0$

$Type_\omega$

$\vec{\omega}_0$

[Advanced Variable Settings](#)

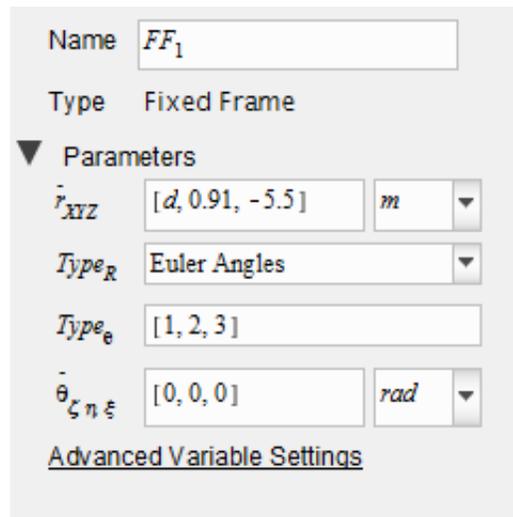
Fig. 4: Rigid Body Parameters

Step 6: Set up the net

This step creates a rod that helps visualize the top of the net in the 3-D view. The length of a tennis net is approximately 11m and the height of the top of the net is approximately 0.91m at the center.

1. Click the **Fixed Frame** component and, as shown in Fig. 5, enter

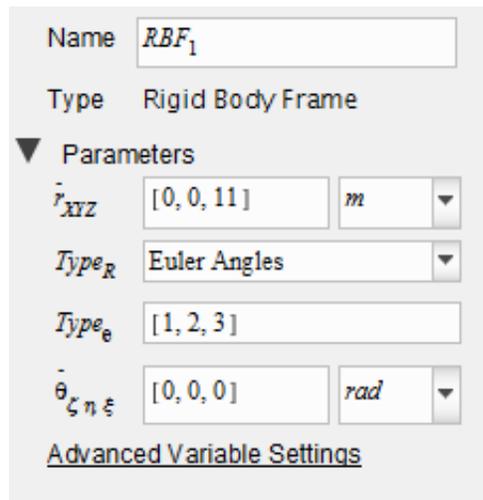
$[d, 0.91, -5.5]$ for the location of the **Fixed Frame**.



The screenshot shows a software interface for defining a Fixed Frame. The 'Name' field contains 'FF₁'. The 'Type' is set to 'Fixed Frame'. Under the 'Parameters' section, the position vector \vec{r}_{XYZ} is [d, 0.91, -5.5] with units 'm'. The rotation type is 'Euler Angles'. The sequence is [1, 2, 3]. The rotation angles $\vec{\theta}_{\zeta \eta \xi}$ are [0, 0, 0] with units 'rad'. A link for 'Advanced Variable Settings' is at the bottom.

Fig. 5: Fixed Frame Parameters

- Then, as shown in Fig. 6, click the **Rigid Body Frame** component and change the length in the z direction to **11m**. This length is not important and does not affect the results of the simulation, it just aids with the visualization.



The screenshot shows a software interface for defining a Rigid Body Frame. The 'Name' field contains 'RBF₁'. The 'Type' is set to 'Rigid Body Frame'. Under the 'Parameters' section, the position vector \vec{r}_{XYZ} is [0, 0, 11] with units 'm'. The rotation type is 'Euler Angles'. The sequence is [1, 2, 3]. The rotation angles $\vec{\theta}_{\zeta \eta \xi}$ are [0, 0, 0] with units 'rad'. A link for 'Advanced Variable Settings' is at the bottom.

Fig. 6: Rigid Body Frame Parameters

Step 7: Add a plot window for the trajectory

1. Attach a **Probe** as shown in Fig. 1.
2. Click the probe and select **1** and **2** in the **Inspector** tab. In this case, 1 corresponds to the x-position and 2 corresponds to the y-position.
3. Using the **Plot** tab, add a Plot window for the trajectory of the ball. To add a window, click the drop down menu that says **Default Plot Window** and click -- **Add Window**--. After giving the window a name, click **OK** and then click **Empty** to show the plot options. Give the plot a title and set up the plot to show **Probe1: value[1]** on the x-axis and **Probe2: value[2]** on the y-axis.

Step 8: Run the simulation

1. Click the **Greater Equal Threshold** component and enter d in the threshold textbox in the **Inspector** tab.
2. Run the simulation for $\theta=0$ (Problem Part a).

The simulation can be rerun by changing the other parameters to see the different combinations of speed, angle, height and distance that allow the ball to clear the net. The following image shows what the 3-D view should look like for Part c) of the problem.

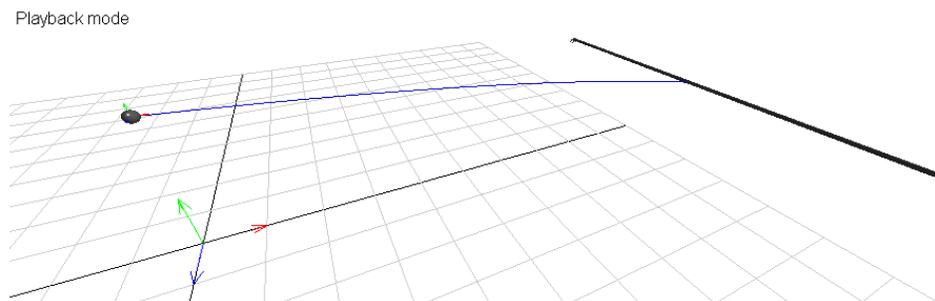


Fig. 7: 3-D view of the tennis ball simulation

Example 2: Target Shooting

Problem Statement: An air rifle that shoots pellets with a speed of 100 m/s is to be aimed at an apple placed 100 m away. The center of the apple is at the same height as the muzzle.

- At what angle above the horizontal must the rifle be pointed so that the pellet hits the apple dead center?
- How high above the center of the apple should the rifle be aimed?
- What is the maximum vertical displacement of the pellet when it follows the required trajectory?
- What is the vertical velocity of the pellet 0.25 sec before it hits the target?

Analytical Solution

restart;

Data:

$$v_0 \quad [m/s]$$

$$:= 10$$

$$d \quad [m]$$

$$:= 10$$

$$g \quad [m/s^2]$$

$$:= 9.81$$

Solution:

Part a) Determining the rifle angle required to hit the target dead center.

Since the muzzle exit and the target are at the same height, for the pellet to hit the center of the target, the vertical displacement of the pellet should be zero when the pellet reaches the plane of the target. By combining Eq. (7) and Eq. (11), and substituting the parameters specific to this problem, the following equation is obtained.

$$0 = (d) \cdot \tan(\theta) - \frac{g \cdot (d)^2}{2(v_0 \cdot \cos(\theta))^2} :$$

$$\theta := \text{solve} \left(0 = (d) \cdot \tan(\theta) - \frac{g \cdot (d)^2}{2(v_0 \cdot \cos(\theta))^2}, \theta \right)$$

$$1.522, .049, -1.620, -3.092$$

(2.2.1.1.1)

Due to the form of the equation, there are many angles that satisfy it. For this case, only the smallest angle in the first quadrant, which is 0.049 rad or 2.81°, is of interest. Therefore, the rifle needs to be pointed at an angle of 2.81° above the horizontal to hit the target dead center.

Part b) Determining the height of the point above the target that has to be aimed at.

If the rifle needs to be pointed at an angle of θ_0 , it needs to be aimed at a point $100 \cdot \tan(\theta_0)$ above the center of the target.

Since,

$$\theta_0 := 0.049 :$$

the height above the target to which the rifle should be pointed is,

$$h_{aim} := 100 \cdot \tan(\theta_0) = 4.904$$

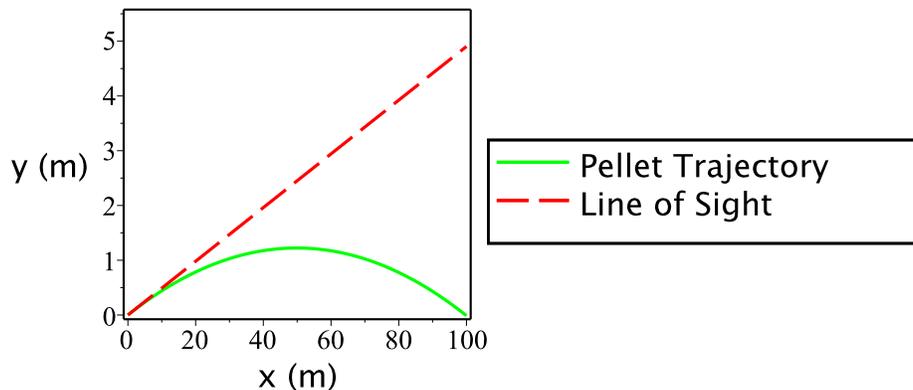
Therefore, the rifle has to aimed at a point 4.9 m above the target for the pellet to hit the target dead center.

The following plot contains the pellet trajectory and the line of sight of the rifle.



Trajectory - Part c).

Pellet Trajectory



Part c) Determining the maximum vertical displacement of the pellet for the required trajectory.

Using Eq. (16), the maximum vertical displacement of the bullet is

$$y := \frac{(v_0 \cdot \sin(\theta_0))^2}{2 \cdot g} = 1.2228$$

Therefore, the maximum vertical displacement of the pellet for the required trajectory is 1.22 m.

Part d) Determining the vertical velocity of the pellet 0.25 sec before impact.

The total time it takes the pellet to strike the target can be found using Eq. (7).

$$t := \frac{v_0 \cdot \cos(\theta_0)}{d} = .9988$$

The vertical velocity 0.25 sec before impact can be calculated using Eq. (13).

$$v_y := v_0 \cdot \sin(\theta_0) - g \cdot (t - 0.25) = -2.4477$$

Therefore, the velocity of the pellet 0.25 seconds before impact is -2.45 m/s.

MapleSim Simulation

Constructing the Model

Step1: Insert Components

Drag the following components into the workspace:

Table 2: Components and locations

Component	Location
 (2 required)	Multibody > Body and Frames
	Multibody > Visualization
	Multibody >

	Sensors
	Signal Blocks > Routing > DeMultiplexers
	Signal Blocks > Relational
	Signal Blocks > Boolean
	Multibody > Body and Frames

Step 2: Connect the components

Connect the components as shown in the following diagram.

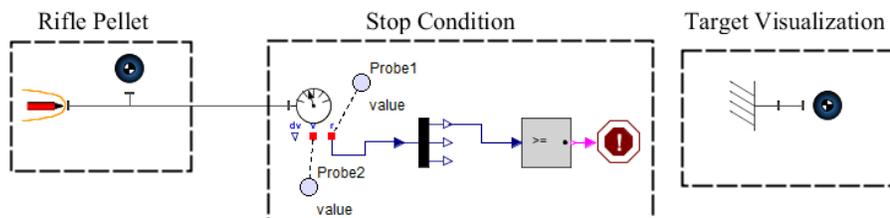


Fig. 8: Component diagram

Step 3: Create parameters

Click the **Add a Parameter Block** icon () , click on the workspace and double click the **Parameters** icon that will appear on the workspace. Create parameters for the initial speed of the ball v , the distance to the target d , and the launch angle $theta$ (as shown below).

Parameters subsystem default settings

Name	Type	Default Value	Default Units	Description
v	Velocity [$\frac{m}{s}$]	1	$\frac{m}{s}$	Initial speed (m/s)
d	Position [m]	1	m	Horizontal distance to the target (m)
theta	Angle [rad]	1	deg	Launch angle measured from the horizontal (deg)

Fig. 9: Parameter Block Settings

Step 4: Adjust the parameters

Return to the main diagram ( > **Main** ►), then click the **Parameters** icon in the workspace. Change the parameters in the **Inspector** tab as shown in Fig 10.

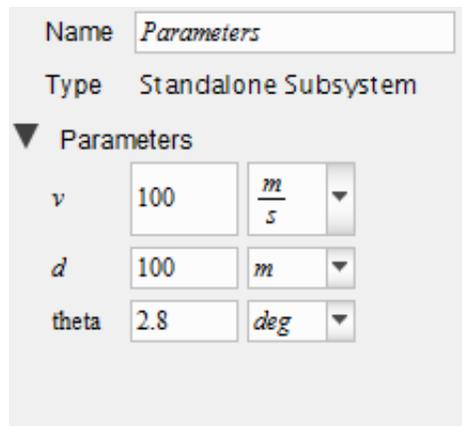


Fig. 10: Parameters

Step 5: Change the initial conditions

Return to the main diagram and click the **Rigid Body** component that corresponds to the rifle pellet. As shown in Fig. 11, enter the following initial conditions (Fig. 11) in the **Inspector** tab in terms of the parameters.

Name

Type Rigid Body

▼ Parameters

m

$[I]$

0.00010	0.0	0.0
0.0	0.00010	0.0
0.0	0.0	0.00010

$IC_{r,v}$

\vec{r}_0

$Type_v$

\vec{v}_0

$IC_{e,\omega}$

$Type_e$

$\vec{\theta}_0$

$Type_\omega$

$\vec{\omega}_0$

[Advanced Variable Settings](#)

Fig. 11: Rigid Body Parameters

Step 6: Set up the target

This step creates another sphere that represents the apple. This helps visualize the target in the 3-D view.

Click the **Fixed Frame** component and change the position of the fixed frame to **[d, 0,0]**.

Step 7: Add a plot window for the trajectory

1. Click **Probe 1** and select **1** and **2** in the **Inspector** tab. In this case, 1 corresponds to the x-position and 2 corresponds to the y-position.
2. Using the **Plot** tab, add a plot window for the trajectory of the pellet. To add a window, click the **Default Plot Window** drop down menu and click **--Add Window--**. After giving the window a name, click **OK** and then click **Empty** to show the plot options. Give the plot a title and set up the plot to show **Probe1: value[1]** on the x-axis and **Probe2: value[2]** on the y-axis.

Step 8: Run the simulation

1. Click **Probe 2** and select **1** and **2** in the **Inspector** tab. Also, click on the **Greater Equal Threshold** component and enter **d** in the threshold textbox in the **Inspector** tab.
2. Click **Run Simulation** (▶).

The following figure shows the 3-D visualization for the simulation.

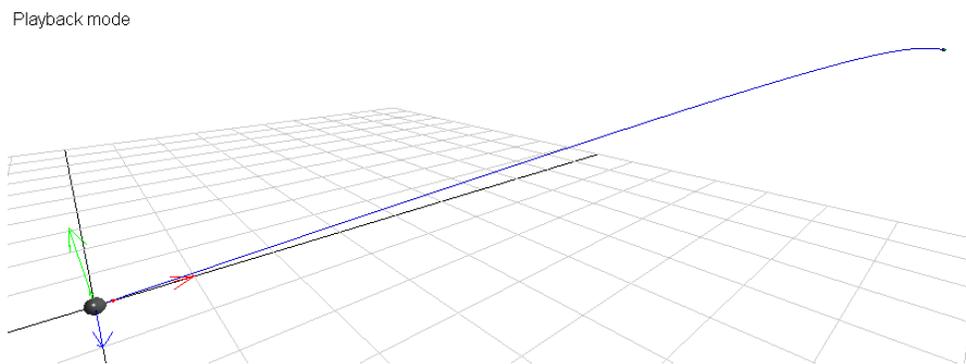


Fig. 12: 3-D visualization

One step further: Incorporating Drag

Drag force refers to the force that opposes the relative motion of an object in a fluid. In the case of the tennis ball, it is resistance offered by the air.

The force due to drag is expressed using the following equation:

$$F_D = \frac{1}{2} \cdot C_D \cdot \rho \cdot A \cdot v^2$$

... Eq. (22)

where, C_D is the coefficient of drag, ρ is the density of the fluid, A is the characteristic area of the object and v is the speed of the object relative to the fluid. In the case of a ball, the characteristic area is the area of cross-section. The coefficient of drag depends on a lot of different factors and is usually very difficult to calculate analytically. Hence the coefficient of drag is usually found through experiments or using numerical methods. For the case of a sphere, coefficient of drag curves are commonly available in textbooks and the internet. Fig. 13 is obtained from the www.nasa.gov website (available at: <http://www.grc.nasa.gov/WWW/k-12/airplane/dragsphere.html>).

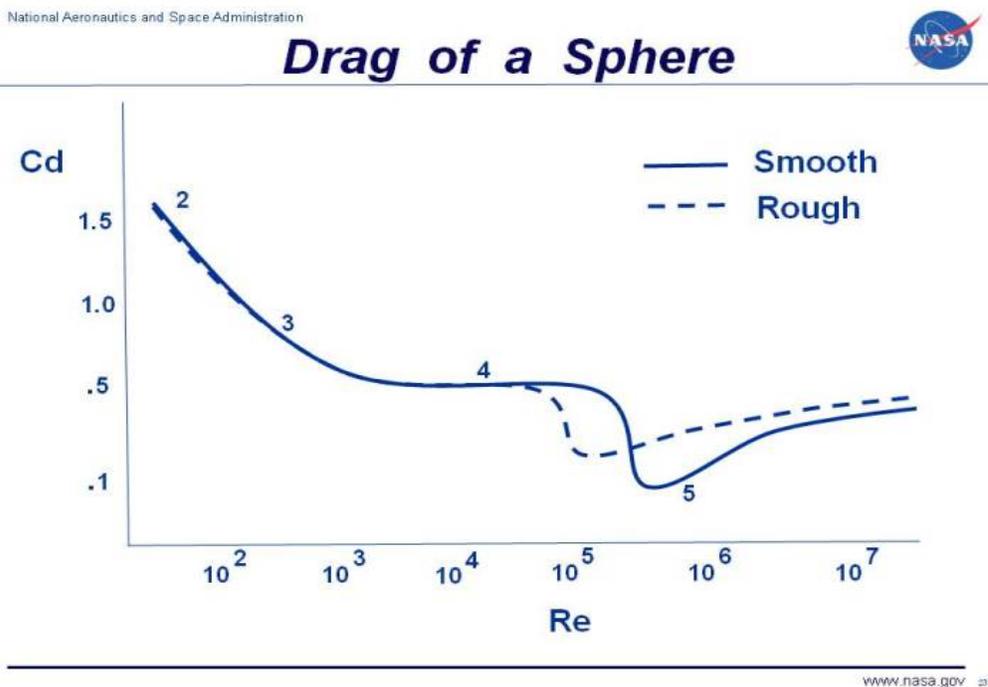


Fig. 13: Coefficient of drag for spheres.

As can be seen, the coefficient of drag is plotted as a function of the Reynolds number Re . This is a dimensionless number which is very commonly used in the field of fluid dynamics. It gives a measure of the ratio of inertial forces to viscous forces. For a sphere the Reynolds

number is expressed as,

$$\text{Re} = \frac{\rho \cdot v \cdot D}{\mu}$$

... Eq. (23)

where, ρ is the density of the fluid, v is the velocity of the flow, D is the diameter of the sphere and μ is the viscosity of the fluid.

If the ball is moving through the air with a velocity of $\vec{v} = v_x \hat{i} + v_y \hat{j}$ then the direction of the drag force will be in a direction opposite to this vector. The magnitude of this drag force will be given by Eq. (22). In vector form this force will be

$$\vec{F} = \frac{1}{2} \cdot C_D \cdot \rho \cdot A \cdot (v_x^2 + v_y^2) \cdot \left[\frac{v_x}{\sqrt{v_x^2 + v_y^2}} \hat{i} + \frac{v_y}{\sqrt{v_x^2 + v_y^2}} \hat{j} \right]$$

... Eq. (24)

This can be simplified and written as,

$$\vec{F} = \frac{1}{2} \cdot C_D \cdot \rho \cdot A \cdot v \cdot [v_x \hat{i} + v_y \hat{j}]$$

... Eq. (25)

where, $v = \sqrt{v_x^2 + v_y^2}$ is the speed of the ball.

This can be incorporated into the MapleSim model as shown in the following diagram. Since the coefficient of drag is not constant, points from the curve are input into a Microsoft Excel spreadsheet in the form of a table and attached to the model. For the case of the tennis ball, the curve that corresponds to a rough sphere has been used. This data can then be used by the **1D Lookup Table** component which interpolates between the points.

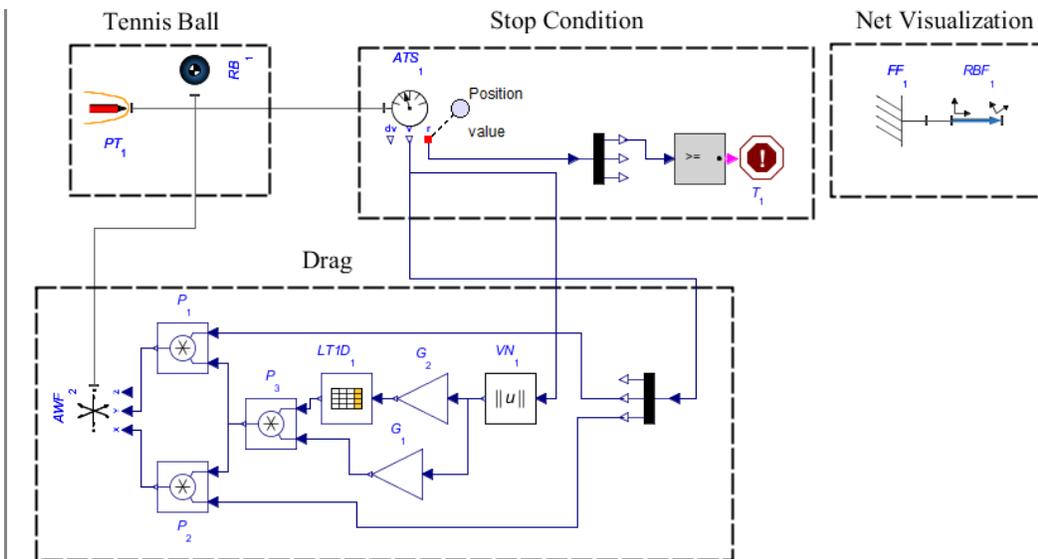


Fig. 14: Component diagram for the Tennis example with drag.

This model can be played around with to see the effect of drag. Its results can then be compared to the previous model that does not include drag to see the differences. For a ball moving at a speed around 30 m/s, the Re number is approximately $1.4 \cdot 10^5$. By looking at the coefficient of drag curve, it can be seen that the coefficient of drag for this Re number is approximately 0.2 which is not very high. It can also be observed that the coefficient of drag values suddenly dips around a Re number of 10^5 for rough spheres. This early reduction of drag for rough spheres plays a significant role in sports like tennis, golf, baseball, cricket, etc.

By running this simulation with theta equal to 0 and $v_0 = 30\text{m/s}$, the ball is predicted to be approximately 1.3 mm lower at the net than the prediction of the model without drag. This shows that neglecting drag in this case is an extremely good assumption.

Reference:

Halliday et al. "Fundamentals of Physics", 7th Edition. 111 River Street, NJ, 2005, John Wiley & Sons, Inc.