

Classroom Tips and Techniques: Electric Field from Distributed Charge

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Introduction

This past summer I was asked if Maple could replicate and complete a solution of the following electrostatics problem.

A line charge of density $\rho_l = 6.0 \text{ nC/m}$ is on the z -axis, $-1.5 < z < 1.5$, in meters. Find $\mathbf{E}(r, \theta, z)$ everywhere in free space; $\mathbf{E}(r, \theta, z)$ is the electric field expressed in cylindrical coordinates.

The solution to be replicated is summarized in Table 1.

Cylindrical symmetry about the z -axis implies the field is determined by its values in a plane, taken, for example, as the xz -plane. With \mathbf{r} as the position vector in \mathbb{R}^3 , and \mathbf{r}' as the location of the charge, the electric field is given by

$$\mathbf{E}(\mathbf{r}) = \int_{\text{line}} \frac{\rho_{l, \mathbf{r}'}(\mathbf{r} - \mathbf{r}')}{4\pi\epsilon |\mathbf{r} - \mathbf{r}'|^3} dL'$$

where L' represents the path of integration along the line of charge. Since the line charge is constant along the line, we have $\rho_{l, \mathbf{r}'} = \rho_l$. The free-space assumption implies $\epsilon = \epsilon_0 = 8.85 \times 10^{-12} \text{ F/m}$. Symmetry implies $\mathbf{r} = x \hat{\mathbf{x}} + z \hat{\mathbf{z}}$.

Hence

$$\mathbf{r} - \mathbf{r}' = (z \hat{\mathbf{x}} + z \hat{\mathbf{z}}) - (z' \hat{\mathbf{z}}) = x \hat{\mathbf{x}} + (z - z') \hat{\mathbf{z}}$$

and

$$|\mathbf{r} - \mathbf{r}'|^2 = x^2 + (z - z')^2$$

With the charge distribution given by $\mathbf{r}' = z \hat{\mathbf{z}}$ for $-1.5 < z < 1.5$, we then have $dL' = dz'$.

Consequently, we have

$$\begin{aligned}
 \mathbf{E}(\mathbf{r}) &= \int_{line} \frac{\rho_l \mathbf{r}'(\mathbf{r} - \mathbf{r}')}{4 \pi \epsilon |\mathbf{r} - \mathbf{r}'|^3} dL' \\
 &= \int_{z'=-1.5}^{z'=1.5} \frac{\rho_l(\mathbf{r} - \mathbf{r}')}{4 \pi \epsilon_0 |\mathbf{r} - \mathbf{r}'|^3} dz' \\
 &= \frac{\rho_l}{4 \pi \epsilon_0} \int_{z'=-1.5}^{z'=1.5} \frac{x \widehat{\mathbf{x}} + (z - z') \widehat{\mathbf{z}}}{(x^2 + (z - z')^2)^{3/2}} dz' \\
 &= \frac{\rho_l}{4 \pi \epsilon_0} \left(\widehat{\mathbf{x}} \int_{z'=-1.5}^{z'=1.5} \frac{x}{(x^2 + (z - z')^2)^{3/2}} dz' + \widehat{\mathbf{z}} \int_{z'=-1.5}^{z'=1.5} \frac{(z - z')}{(x^2 + (z - z')^2)^{3/2}} dz' \right)
 \end{aligned}$$

Table 1 Typewritten solution of given electrostatics problem

In the following sections, Maple solutions of this problem are given, one using the *VectorCalculus* package, and one using the *Physics* package. In each case, the integrals giving the electric field are evaluated. The electric field vectors are graphed, and field lines are calculated and displayed. Thus, the ability of Maple to enhance a pedagogical experience is amply manifest.

Maple Solutions

Solution in the *VectorCalculus* Package

```

> with(VectorCalculus) :
with(plots) :
with(DEtools) :
BasisFormat(false) :
>

```

Use of the *VectorCalculus* package has three advantages. First, you can control the appearance of vectors with the **BasisFormat** command. The default setting causes vectors to be displayed as linear combinations of basis vectors. The setting chosen here causes vectors to be displayed

as column vectors. To make the change, change "true" to "false" in the **BasisFormat** command.

Second, the **Norm** command defaults to the Euclidean norm and all quantities are treated as real. At top level, the vectors would also appear as columns, but the available vector norm would treat quantities as complex, and complex conjugates and absolute values would then appear.

Third, the **int** command is modified. At top level, **int** represents integration in a single variable, and does not automatically map to the components of a vector. In *VectorCalculus*, **int** can be used to represent iterated (multiple) integrals, and also maps automatically to vectors.

Because Maple understands the prime as a differentiation operator, it's a lot of work to over-ride the prime and use it strictly as notation. Hence, **R** will be used for r' , and Z for z' . With these conventions, make the following definitions.

> $\mathbf{r} := \langle x, 0, z \rangle :$

$\mathbf{R} := \langle 0, 0, Z \rangle :$

$\mathbf{d} := \mathbf{r} - \mathbf{R} :$

$\mathbf{r}, \mathbf{R}, \mathbf{d}$

$$\begin{bmatrix} x \\ 0 \\ z \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ Z \end{bmatrix}, \begin{bmatrix} x \\ 0 \\ z - Z \end{bmatrix}$$

>

The integrand will then be

> $\mathbf{f} := \frac{\mathbf{d}}{\|\mathbf{d}\|^3}$

$$f := \begin{bmatrix} \frac{x}{(x^2 + z^2 - 2zZ + Z^2)^{3/2}} \\ 0 \\ \frac{z - Z}{(x^2 + z^2 - 2zZ + Z^2)^{3/2}} \end{bmatrix}$$

>

The electric field vector is then

> $\mathbf{E} := \int_{-3/2}^{3/2} \mathbf{f} dZ \text{ assuming } positive$

$$E := \left[\left[\begin{aligned} & - \left(-2 \sqrt{4x^2 + 4z^2 - 12z + 9} z \right. \right. \\ & \left. \left. - 3 \sqrt{4x^2 + 4z^2 - 12z + 9} + 2 \sqrt{4x^2 + 4z^2 + 12z + 9} z \right. \right. \\ & \left. \left. - 3 \sqrt{4x^2 + 4z^2 + 12z + 9} \right) / \right. \\ & \left. \left(\sqrt{4x^2 + 4z^2 + 12z + 9} x \sqrt{4x^2 + 4z^2 - 12z + 9} \right) \right], \\ & \left[0 \right], \\ & \left[- \frac{2 \left(\sqrt{4x^2 + 4z^2 - 12z + 9} - \sqrt{4x^2 + 4z^2 + 12z + 9} \right)}{\sqrt{4x^2 + 4z^2 + 12z + 9} \sqrt{4x^2 + 4z^2 - 12z + 9}} \right] \end{aligned} \right]$$

>

Notice that Maple has evaluated the integral and returned the electric field as a vector.

Setting $y=0$ and calling x by the name r gives the cylindrically symmetric solution obtained in Table 1.

> **EE** := **E** $\Big|_{x=r, y=0}$

$$EE := \left[\left[\begin{aligned} & - \left(-2 \sqrt{4r^2 + 4z^2 - 12z + 9} z \right. \right. \\ & \left. \left. - 3 \sqrt{4r^2 + 4z^2 - 12z + 9} + 2 \sqrt{4r^2 + 4z^2 + 12z + 9} z \right. \right. \\ & \left. \left. - 3 \sqrt{4r^2 + 4z^2 + 12z + 9} \right) / \right. \\ & \left. \left(\sqrt{4r^2 + 4z^2 + 12z + 9} r \sqrt{4r^2 + 4z^2 - 12z + 9} \right) \right], \\ & \left[0 \right], \\ & \left[- \frac{2 \left(\sqrt{4r^2 + 4z^2 - 12z + 9} - \sqrt{4r^2 + 4z^2 + 12z + 9} \right)}{\sqrt{4r^2 + 4z^2 + 12z + 9} \sqrt{4r^2 + 4z^2 - 12z + 9}} \right] \end{aligned} \right]$$

>

The reader should pay particular attention to a notational issue not obvious without comment. The name \mathbf{r} has been assigned a vector as its value. Ordinarily, this would persist in the substitution made into E . To avoid this clash, we have converted r to an "atomic identifier" via Context Menu: 2-D Math>Convert To>Atomic Identifier. The symbol r is now seen by Maple as a name distinct from \mathbf{r} .

>

Graphics

Figure 1 provides a graph of the electric field as a collection of arrows.

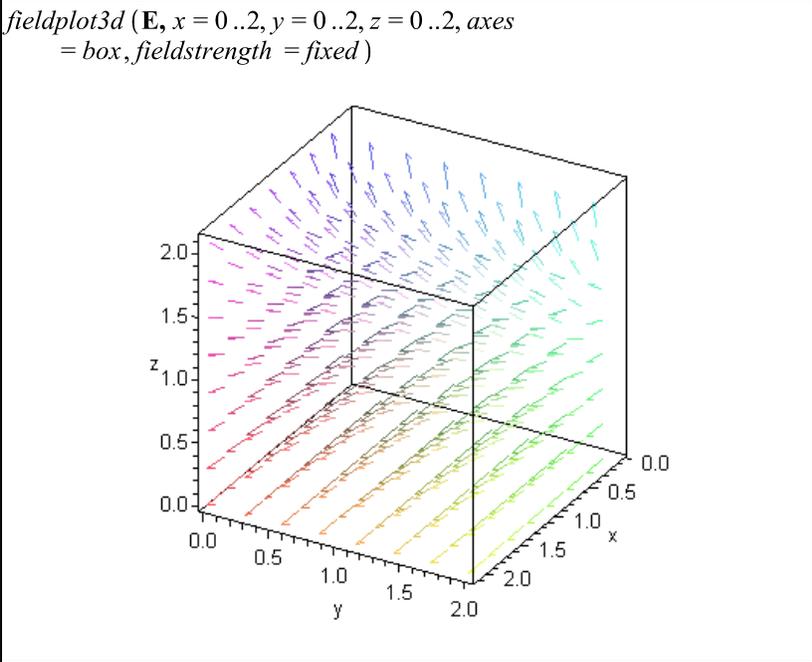


Figure 1 The electric field in \mathbb{R}^3

Figure 2 shows the electric field restricted to the rz -plane.

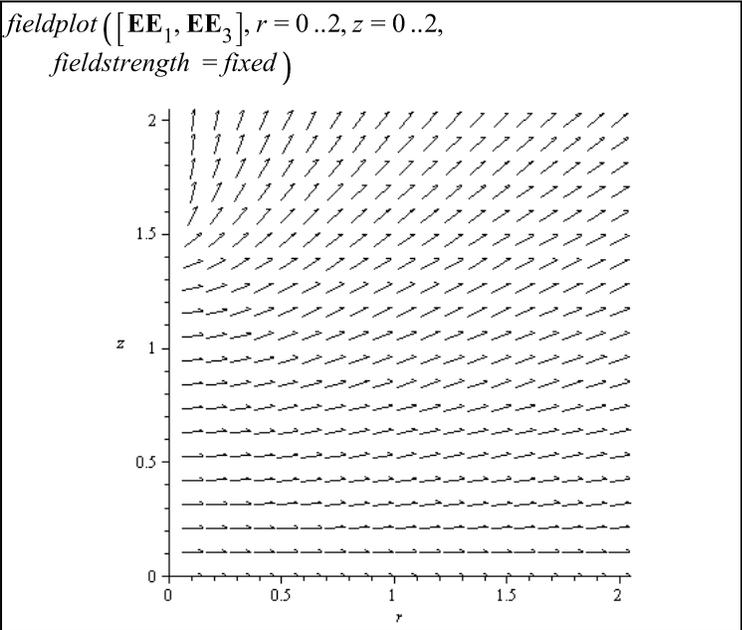


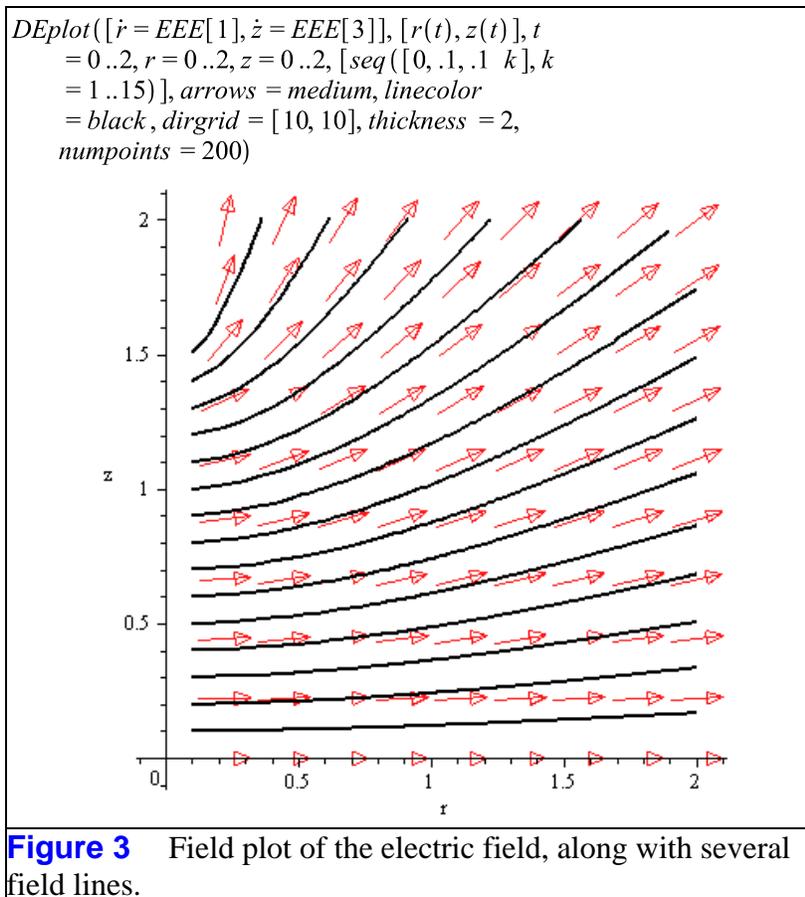
Figure 2 The electric field drawn in the rz -plane

To obtain field lines, solve the differential equations $\frac{d}{dt} \begin{bmatrix} r \\ z \end{bmatrix} = \mathbf{EE}$. The dependent variables in \mathbf{EE} must explicitly be functions of the independent variable t . Thus, write

```
> EEE := EE |  
      |r = r(t), z = z(t) :
```

```
>
```

then obtain numeric solutions and a graph with the **DEplot** command, as shown in Figure 3.



```
>
```

Solution in the *Physics* Package

An alternative solution in the *Physics* package displays the unit basis vectors as $\{\widehat{i}, \widehat{j}, \widehat{k}\}$, and vectors that are linear combinations of these vectors as symbols with an arrow superimposed.

```
> restart
```

```
> with(Physics:-Vectors) :
```

```
Setup(mathematicalnotation = true) :
```

```
>
```

Define the position vector as \vec{r} and the vector \mathbf{r}' as \vec{R} by typing

```
> r_ := x*_i + z*_k;
R_ := Z*_k;
>
```

and converting this text input to 2-D Math Input via the Format>Convert To menu options. The result will be

```
>  $\vec{r} := x \hat{i} + z \hat{k}$ 
 $\vec{R} := Z \hat{k}$ 
 $\vec{d} := \vec{r} - \vec{R}$ 
```

$$\vec{r} := x \hat{i} + z \hat{k}$$

$$\vec{R} := Z \hat{k}$$

$$\vec{d} := x \hat{i} + \hat{k} (z - Z)$$

>
so that the integrand is

```
>  $\vec{f} := \frac{\vec{d}}{(\vec{d} \cdot \vec{d})^{3/2}}$ 
```

$$\vec{f} := \frac{x \hat{i} + \hat{k} (z - Z)}{(x^2 + (z - Z)^2)^{3/2}}$$

>
The electric field vector is then

```
>  $\vec{E} := \int_{-\frac{3}{2}}^{\frac{3}{2}} \vec{f} dZ$ 
```

$$\vec{E} := \left(2\sqrt{4x^2 + 4z^2 - 12z + 9} \hat{i} z - 2\sqrt{4x^2 + 4z^2 - 12z + 9} \hat{k} x + 3\sqrt{4x^2 + 4z^2 - 12z + 9} \hat{i} - 2\sqrt{4x^2 + 4z^2 + 12z + 9} \hat{i} z + 2\sqrt{4x^2 + 4z^2 + 12z + 9} \hat{k} x + 3\sqrt{4x^2 + 4z^2 + 12z + 9} \hat{i} \right) / \left(\sqrt{4x^2 + 4z^2 + 12z + 9} x \sqrt{4x^2 + 4z^2 - 12z + 9} \right)$$

>
The components of the electric field vector can be extracted as

```
> E1 := Component( $\vec{E}$ , 1)
```

$$E1 := \frac{2z}{\sqrt{4x^2 + 4z^2 + 12z + 9} x} + \frac{3}{\sqrt{4x^2 + 4z^2 + 12z + 9} x}$$

$$- \frac{2z}{x \sqrt{4x^2 + 4z^2 - 12z + 9}} + \frac{3}{x \sqrt{4x^2 + 4z^2 - 12z + 9}}$$

>
and

> $E3 := \text{Component}(\vec{E}, 3)$

$$E3 := -\frac{2}{\sqrt{4x^2 + 4z^2 + 12z + 9}} + \frac{2}{\sqrt{4x^2 + 4z^2 - 12z + 9}}$$

>

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