

Classroom Tips and Techniques: A Vexing Trig Conversion

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Introduction

In engineering, it is common to express the sum

$$a e^{c t} \cos(\omega t) + b e^{c t} \sin(\omega t)$$

in the form

$$A e^{c t} \cos(\omega t - \phi)$$

where $A = \sqrt{a^2 + b^2}$ and $\phi = \arctan\left(\frac{b}{a}\right)$. Expressions like this often arise in the solution of initial value problems for damped oscillators, so the transformation is really useful for extracting the magnitude and phase shift of the oscillations. Unfortunately, Maple does not presently provide a built-in facility for making this transformation.

In [1], I explored use of Maple's **solve/identity** construct to effect the transformation in Maple V Release 4, the extant version of Maple at that time (1996). A year later in [2], Dr. Michael Monagan followed up on this question, and provided a procedure for the transformation, and applied it to

$$y(t) = e^{t/2} \cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{\sqrt{3}}{3} e^{t/2} \sin\left(\frac{\sqrt{3}}{2}t\right)$$

the solution of the IVP $\ddot{y} - \dot{y} + y = 0, y(0) = \dot{y}(0) = 1$, obtaining

$$y(t) = \frac{2\sqrt{3}}{3} e^{t/2} \sin\left(\frac{\sqrt{3}}{2}t + \frac{\pi}{3}\right)$$

It's not clear from the article which version of Maple was used, but the conclusion, that Maple converted the cosine to a sine resulted in his writing

"Uggghh! Maple has simplified the cosine to a sine on us. It has made the transformation $\cos\left(x - \frac{\pi}{6}\right) = \sin\left(x + \frac{\pi}{3}\right)$ thinking that we'd prefer the result with a

positive angle."

We revisit both approaches, then add another alternative to the mix.

Initializations

restart

The 1996 solve/identity Approach

To convert

$$5 e^{-3 t} \cos(4 t) - 9 e^{-3 t} \sin(4 t)$$

to the form

$$A e^{-3 t} \cos(4 t - z)$$

use the Maple command

`solve(identity((3.1) = (3.2), t), {A, z})`

$$\left\{ A = \sqrt{106}, z = -\arctan\left(\frac{9}{5}\right) \right\}, \left\{ A = -\sqrt{106}, z = -\arctan\left(\frac{9}{5}\right) + \pi \right\}$$

The solution with positive amplitude A is the desired one, and results in the form

(3.2) |
(3.3) [1]

$$\sqrt{106} e^{-3 t} \cos\left(4 t + \arctan\left(\frac{9}{5}\right)\right)$$

This is essentially the same approach as in [1], but no longer is it necessary to expand (3.2) with all the concomitant difficulties of keeping $\cos(4 t)$ and $\sin(4 t)$ from themselves expanding in powers of $\sin(t)$ and $\cos(t)$.

As an update to the calculation, I would like Maple to determine the coefficient of t in the exponential, and w , the value of angular frequency in the cosine and sine terms. To this end, try to match against a more general template via the command

`solve(identity((3.1) = A e^{c t} cos(\omega t - \phi), t), {A, c, \omega, \phi})`

Warning, solutions may have been lost

Unfortunately, this does not work, and even setting the environment variable `_EnvAllSolutions` to true does not change the outcome. Hence, we revisit Monagan's code to see if it is more robust than the `solve/identity` device.

Monagan's 1997 Code

In [2], Monagan goes through several iterations of a procedure designed to morph (3.1) to (3.2), thinking out loud as to why he makes the various modifications shown. The result is the pair of procedures given in Table 1, where the second procedure invokes the first.

<pre> common := proc(f, g, newf, newg) local C, F, G; if type(f, `*`) then F := {op(f)}; else F := {f}; end if; if type(g, `*`) then G := {op(g)}; else G := {g}; end if; C := F intersect G; newf := *(op(F minus C)); newg := *(op(G minus C)); return *(op(C)); end proc: </pre>	<pre> trigcombine := proc(f, x) local A, B, s, C, phi; if type(f, {name, numeric, procedure }) then f; elif type(f, `+`) and degree(f, sin(x)) = 1 and degree(f, cos(x)) = 1 and degree(f, {sin(x), cos(x)}) = 1 then A := coeff(f, sin(x)); B := coeff(f, cos(x)); s := common(A, B, A', B'); C := sqrt(A^2 + B^2); phi := arctan(-A, B); f - A s sin(x) - B s cos(x) + s C cos(x + phi); else map(trigcombine, f, x); end if; end proc: </pre>
<p>Table 1 Monagan's procedures from [2] that morph (3.1) to (3.2)</p>	

Applying the **trigcombine** procedure, we have

trigcombine((3.1), 4 t)

$$\sqrt{106} e^{-3t} \cos\left(4t + \arctan\left(\frac{9}{5}\right)\right)$$

The earlier version of Maple used for [2] returned this as $\frac{2\sqrt{3}}{3} e^{t/2} \sin\left(\frac{\sqrt{3}}{2}t + \frac{\pi}{3}\right)$, prompting Monagan's Ughhhh! However, even though Maple 12 now returns the cosine, as expected, one still has to include the extra argument, 4 t. In return, we note that this procedure is robust enough to convert the generic

$$\alpha e^{\gamma t} \cos(\omega t) + \beta e^{\gamma t} \sin(\omega t)$$

to

trigcombine((4.2), ω t)

$$e^{\gamma t} \sqrt{\alpha^2 + \beta^2} \cos(\omega t + \arctan(-\beta, \alpha))$$

Maple 12 in 2008

Monagan's code in the right-hand column of Table 1 uses the **coeff** command to obtain the coefficients of the trig terms. However, for this to work, the trig terms must have arguments, which is why the **trigcombine** command requires the second argument, ωt . We sought code that would deduce this argument. Dr. James McCarron, one of the Maple developers, provided that code and a lot more in the procedure **TrigCombine**, defined in the Code Edit Region below.

To access the procedure **TrigCombine**, click the button



To view the code in this Code Edit Region, use Context Menu: Expand Code Edit Region. The code is work worthy of one of the Maple developers. It finds the appropriate trig terms in larger expressions, and applies the requisite transformation in every instance possible in the given expression.

The **TrigCombine** procedure evolved from the desire to write

$$-e^{-\frac{1}{4}t} \cos\left(\frac{1}{4}t\sqrt{15}\right) - \frac{1}{15} e^{-\frac{1}{4}t} \sin\left(\frac{1}{4}t\sqrt{15}\right) \sqrt{15} + 1$$

in the form

$$1 - \frac{4}{15} \sqrt{15} e^{-t/4} \cos\left(\frac{1}{4}t\sqrt{15} - \arctan\left(\frac{1}{15}\sqrt{15}\right)\right)$$

without having to retype the argument of the trig functions. Monagan's **trigcombine** returns unevaluated because of the "+1" in (5.1). Application of the **TrigCombine** procedure then gives

TrigCombine((5.1), t)

$$1 - \frac{4}{15} \sqrt{15} e^{-\frac{1}{4}t} \cos\left(\frac{1}{4}t\sqrt{15} - \arctan\left(\frac{1}{15}\sqrt{15}\right)\right)$$

Like Monagan's **trigcombine** procedure, **TrigCombine** is robust enough to work with parameters, as we see from

TrigCombine((4.2), t)

$$\sqrt{\alpha^2 + \beta^2} e^{\gamma t} \cos(\omega t - \arctan(\beta, \alpha))$$

It is even robust enough to find multiple instances of trig terms that need to be combined. For example, if we apply **TrigCombine** to an expression such as

$$A + 3 t^2 e^{-c_1 t} \cos(\omega_1 t) + 4 t^2 e^{-c_1 t} \sin(\omega_1 t) + 5 t^3 e^{-c_2 t} \cos(\omega_2 t) - 7 t^3 e^{-c_2 t} \sin(\omega_2 t)$$

we would get

`TrigCombine((5.2), t)`

$$A + \sqrt{74} t^3 e^{-c_2 t} \cos\left(\omega_2 t + \arctan\left(\frac{7}{5}\right)\right) + 5 t^2 e^{-c_1 t} \cos\left(\omega_1 t - \arctan\left(\frac{4}{3}\right)\right)$$

in distinction from Monagan's **trigcombine** which would return unevaluated on (5.2). It is not an inconvenience to include the independent variable as an argument to the **TrigCombine** command. However, it is anticipated that in a future release of Maple, this functionality will be realized as an option to the **convert** command, with syntax something like

$$\text{convert}(\text{expression}, \text{phaseamp}, t)$$

where the option "phaseamp" refers to the phase-amplitude form of the trig expressions we have been examining.

Monagan's "Ugghhh"

In [2], Monagan expressed great disappointment when Maple returned $\sin\left(x + \frac{\pi}{3}\right)$ where he expected $\cos\left(x - \frac{\pi}{6}\right)$. In fact, entering the latter will always result in the former, as we see from

$$\cos\left(x - \frac{\pi}{6}\right) \qquad \sin\left(x + \frac{1}{3} \pi\right)$$

This is not the fault of Monagan's code, and it will not be the fault of the code in **TrigCombine**, where this conversion will continue to occur. The conversion of the cosine to the sine has been and still is, an automatic simplification in Maple that cannot be blocked by the user. Hence, application of **TrigCombine** to

$$\sqrt{3} \cos(x) + \sin(x)$$

necessarily results in

TrigCombine((6.1), x)

$$2 \sin\left(x + \frac{1}{3} \pi\right)$$

and not in $2 \cos\left(x - \frac{\pi}{6}\right)$.

References

[1] Tips for Maple Instructors, Robert J. Lopez, *MapleTech*, Vol. 3, No. 3, 1966, p. 41.

[2] Tips for Maple Users and Programmers, Michael B. Monagan, *MapleTech*, Vol. 4, No. 3, 1997, p. 13.

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