Hankel Matrix

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Abstract

Using MAPLE 11, properties of the Hankel Matrix have been discussed and some representations have been proposed.

Keywords: Hankel form and matrix; principal minors; forming of Hankel matrices

Definitions

```
> restart:

> with(LinearAlgebra):
```

We consider the following sequence of 2n-1 numbers

```
> seq(a[n],n=0..3),__,a[2*n-2];
```

$$a_0, a_1, a_2, a_3, \_\_, a_{2n-2}$$

which may be the coefficients of a quadratic form in n variables:

```
> Q(x,x):=sum(sum(a[i+k]*x[i]*x[k],i=0..n-1),k=0..n-1);
```

$$Q(x, x) := \sum_{k=0}^{n-1} \left(\sum_{i=0}^{n-1} a_{i+k} x_i x_k \right)$$

This is called a *Hankel form.* The **symmetric** matrix

```
> Matrix([a[i+k],i=0..n-1,k=0..n-1]);
```

$$\left[a_{i+k} \;\; i = 0..n-1 \;\; k = 0..n-1 \right]$$

corresponding to that form is called a *Hankel matrix.* It can be written as:

```
> A[Hankel]:=Matrix([[seq(a[n],n=0..3),__,a[N-1]],
[seq(a[n],n=1..4),__,a[N]],
[seq(a[n],n=2..5),__,a[N+1]],
[seq(a[n],n=3..6),__,a[N+2]],
[°°°,°°°,°°°,°°°,°°°,°°°],
[a[N-1],a[N],a[N+1],a[N+2],__,a[2*N-2]]]);
```

$$A_{Hankel} := \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \_\_ & a_{N-1} \\ a_1 & a_2 & a_3 & a_4 & \_\_ & a_N \\ a_2 & a_3 & a_4 & a_5 & \_\_ & a_{N+1} \\ a_3 & a_4 & a_5 & a_6 & \_\_ & a_{N+2} \\ \circ\circ\circ & \circ\circ\circ & \circ\circ\circ & \circ\circ\circ & \circ\circ\circ & \circ\circ\circ \\ a_{N-1} & a_N & a_{N+1} & a_{N+2} & \_\_ & a_{2N-2} \end{bmatrix}$$

Principal Minors

The sequence of *principal minors* of the Hankel matrix is denoted by

```
> seq(D[q],q=1..3),__,D[N];
```

$$\mathrm{D}_1, \mathrm{D}_2, \mathrm{D}_3, \_\_, \mathrm{D}_N$$

If the first j rows of the Hankel matrix are linear independent, but the first j+1 rows linear dependent, then

```
> D[j]=not_equal_to_zero;
```

$$\mathrm{D}_j = not\_equal\_to\_zero$$

Examples:

```
> Hankel[3*columns]:=HankelMatrix([seq(a[n],n=0..4)]);
```

$$Hankel_{3\ columns} := \begin{bmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{bmatrix}$$

```
> for i in [1,2,3] do
principal_minor[i,i]:=Minor(Hankel[3*columns],i,i,
output=['matrix','determinant'],method='minor')
od;
```

$$principal\_minor_{1,\,1} := \begin{bmatrix} a_2 & a_3 \\ a_3 & a_4 \end{bmatrix},\ a_2\, a_4 - a_3^2$$

$$principal\_minor_{2,\,2} := \begin{bmatrix} a_0 & a_2 \\ a_2 & a_4 \end{bmatrix},\ a_0\, a_4 - a_2^2$$

$$principal\_minor_{3,\,3} := \begin{bmatrix} a_0 & a_1 \\ a_1 & a_2 \end{bmatrix},\ a_0\, a_2 - a_1^2$$

```
> Hankel[4*rows]:=HankelMatrix([seq(a[n],n=0..6)]);
```

$$Hankel_{4\,rows} := \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \\ a_3 & a_4 & a_5 & a_6 \end{bmatrix}$$

```
> for i in [1,2,3,4] do
principal_minor[i,i]:=Minor(Hankel[4*rows],i,i,
output=['matrix','determinant'],method='minor')
od;
```

$$principal\_minor_{1,\,1} := \begin{bmatrix} a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \\ a_4 & a_5 & a_6 \end{bmatrix}, a_2\, a_4\, a_6 - a_2\, a_5^2 + 2\, a_3\, a_5\, a_4 - a_3^2\, a_6 - a_4^3$$

$$principal\_minor_{2,\,2} := \begin{bmatrix} a_0 & a_2 & a_3 \\ a_2 & a_4 & a_5 \\ a_3 & a_5 & a_6 \end{bmatrix}, a_0\, a_4\, a_6 - a_0\, a_5^2 + 2\, a_2\, a_3\, a_5 - a_2^2\, a_6 - a_4\, a_3^2$$

$$principal\_minor_{3,\,3} := \begin{bmatrix} a_0 & a_1 & a_3 \\ a_1 & a_2 & a_4 \\ a_3 & a_4 & a_6 \end{bmatrix}, a_0\, a_2\, a_6 - a_0\, a_4^2 + 2\, a_1\, a_4\, a_3 - a_1^2\, a_6 - a_2\, a_3^2$$

$$principal\_minor_{4,\,4} := \begin{bmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{bmatrix}, a_0\, a_2\, a_4 - a_0\, a_3^2 + 2\, a_1\, a_2\, a_3 - a_1^2\, a_4 - a_2^3$$

The Hankel Matrix formed from the first j Rows

The first j+1 rows of the Hankel matrix are denoted by

```
> seq(R[q],q=1..3),__,R[j+1];
```

$$R_1, R_2, R_3, \_\_, R_{j+1}$$

where

```
> seq(R[q],q=1..3),__,R[j];
```

$$R_1, R_2, R_3, \_\_, R_j$$

are linearly indepedent and R[j+1] is expressed linearly in terms of them:

```
> R[j+1]:=sum(c[p]*R[j-p+1],p=1..j);
```

$$R_{j+1} := \sum_{p=1}^{j} c_p R_{j-p+1}$$

The matrix formed from the first j rows is:

```
> M[R[1]..R[j]]:=Matrix([[a[0],a[1],a[2],__,a[N-1]],
[a[1],a[2],a[3],__,a[N]],
[°°°,°°°,°°°,°°°,°°°],
[a[j-1],a[j],a[j+1],__,a[j+N-2]]]);
```

$$M_{R_1..R_j} := \begin{bmatrix} a_0 & a_1 & a_2 & \_\_ & a_{N-1} \\ a_1 & a_2 & a_3 & \_\_ & a_N \\ \circ\circ\circ & \circ\circ\circ & \circ\circ\circ & \circ\circ\circ & \circ\circ\circ \\ a_{j-1} & a_j & a_{j+1} & \_\_ & a_{j+N-2} \end{bmatrix}$$

This matrix is of rank J < N.

Construction of Hankel Matrices

Hankel matrices are **symmetric** and can be constructed, for instance, in the following ways:

```
> restart:
> with(LinearAlgebra):
> HANKEL[N=1..5]:=seq(Matrix(N,N,(i,k)->a[i-1+k-
1]),N=1..5);
```

$$HANKEL_{N=1..5} := \begin{bmatrix} a_0 \end{bmatrix}, \begin{bmatrix} a_0 & a_1 \\ a_1 & a_2 \end{bmatrix}, \begin{bmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{bmatrix},$$

$$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \\ a_3 & a_4 & a_5 & a_6 \end{bmatrix}, \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ a_1 & a_2 & a_3 & a_4 & a_5 \\ a_2 & a_3 & a_4 & a_5 & a_6 \\ a_3 & a_4 & a_5 & a_6 & a_7 \\ a_4 & a_5 & a_6 & a_7 & a_8 \end{bmatrix}$$

where N is the number of columns or row. There are an odd number of p = 2*N-1 different elements.

```
> restart:
> with(LinearAlgebra):
> for p in [1,3,5,7,9] do
Hankel_matrix[p*different_elements]:=
HankelMatrix([seq(a[n],n=0..p-1)])
```

```
od;
```

$$Hankel\_matrix_{different\_elements} := \begin{bmatrix} a_0 \end{bmatrix}$$

$$Hankel\_matrix_{3\ different\_elements} := \begin{bmatrix} a_0 & a_1 \\ a_1 & a_2 \end{bmatrix}$$

$$Hankel\_matrix_{5\ different\_elements} := \begin{bmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{bmatrix}$$

$$Hankel\_matrix_{7\ different\_elements} := \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \\ a_3 & a_4 & a_5 & a_6 \end{bmatrix}$$

$$Hankel\_matrix_{9\ different\_elements} := \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ a_1 & a_2 & a_3 & a_4 & a_5 \\ a_2 & a_3 & a_4 & a_5 & a_6 \\ a_3 & a_4 & a_5 & a_6 & a_7 \\ a_4 & a_5 & a_6 & a_7 & a_8 \end{bmatrix}$$

These matrices are of rank N, if p = 2*N-1 is the odd number of different elements:

```
> for p in [1,3,5,7,9] do
rank[p*different_elements]:=
Rank(HankelMatrix([seq(a[n],n=0..p-1)]))
od;
```

$$rank_{different\_elements} := 1$$

$$rank_{3\ different\_elements} := 2$$

$$rank_{5\ different\_elements} := 3$$

$$rank_{7\ different\_elements} := 4$$

$$rank_{9\ different\_elements} := 5$$

Further properties of the Hankel Matrix and other forms have been discussed in:

Gantmacher, F.R.: The Theory of Matrices, Volume I and II, Chelsea Publishing Company, New York 1977.