

Classroom Tips and Techniques: Eigenvalue Problems for ODEs

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▼ Introduction

Some boundary value problems for partial differential equations are amenable to analytic techniques. For example, the constant-coefficient, second-order linear equations called the heat, wave, and potential equations are solved with some type of Fourier series representation obtained from the Sturm-Liouville eigenvalue problem that arises upon separating variables.

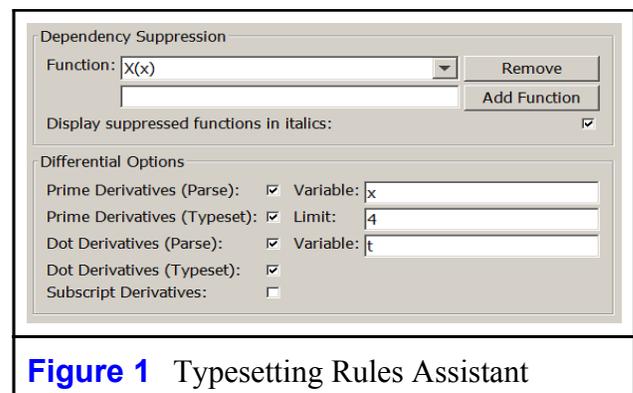
The role of Maple in the solution of such boundary value problems is examined. Efficient techniques for separating variables, and a way to guide Maple through the solution of the resulting Sturm-Liouville eigenvalue problems are shown.

This is the second edition of a May 2007 article originally written with Maple 11. It necessarily made ample use of Maple commands. This present version presents the same material, only in a syntax-free mode. The only exception is the use of the [pdsolve](#) command, for which no Context Menu access is yet available.

▼ Initializations

To gain some notational enhancements, select "Typesetting Rules..." from the View menu, and adjust the settings in the lower-left portion to match those shown in Figure 1.

In the space to the left of the "Add Function" button, enter $X(x)$ and press that button. Then enter $T(t)$ and again press the "Add Function" button. (The drop-down to the left of the "Remove" button allows for removing dependency suppression from a function.) Finally, press the "Done" button in the bottom-right portion of the Assistant.



Alternatively, these settings can be implemented by the following commands.

Typesetting:-Suppress({X(x), T(t)}) :
Typesetting:-Settings(typesetprime = true, typesetdot = true) :

▼ Separating Variables in a PDE

Maple's **pdsolve** command provides, as an eigenfunction expansion, the solution to the following BVP for the one-dimensional wave equation.

$$u_{tt} = c^2 u_{xx}, u(u,) = u(\pi, t) = 0, u(x, 0) = f(x), u_t(x, 0) = 0$$

$$u(x, t) = \sum_{n=1}^{\infty} \frac{2 \left(\int_0^{\pi} f(x) \sin(nx) dx \right) \sin(nx) \cos(cnt)}{\pi}$$

Of course, the eigenfunctions are obtained by separating variables (assume $u(x, t) = X(x) T(t)$) and solving the Sturm-Liouville BVP problem $X'' - \lambda X = 0, X(0) = X(\pi) = 0$ that results. Table 1 summarizes the Maple manipulations that will separate variables in the one-dimensional wave equation.

The wave equation	
<ul style="list-style-type: none"> • Calculus palette: Partial differential operators • Context Menu: Assign to a Name>PDE 	$\frac{\partial^2}{\partial t^2} u(x, t) = c^2 \frac{\partial^2}{\partial x^2} u(x, t) \xrightarrow{\text{assign to a name}} PDE$
Define a separated solution	
<ul style="list-style-type: none"> • Context Menu: Assign to a Name>U 	$X(x) T(t) \xrightarrow{\text{assign to a name}} U$
Separate variables	
<ul style="list-style-type: none"> • Expression palette: Evaluation template Press the Enter key. • Divide by $c^2 U$ and press the Enter key. 	$\begin{aligned} PDE \Big u(x, t) = U \\ X \ddot{T} = c^2 X'' T \end{aligned} \quad (1)$ $\frac{(1)}{c^2 U} \quad \frac{\ddot{T}}{c^2 T} = \frac{X''}{X} \quad (2)$
Extract the two ODEs formed by introducing the Bernoulli separation constant λ	

<ul style="list-style-type: none"> • Reference $\frac{\ddot{T}}{c^2 T} = \frac{X''}{X}$ by its equation label; Press the Enter key. • Context Menu: Left-hand Side • Equate to λ and multiply by $c^2 T$. • Context Menu: Move to Left 	<ul style="list-style-type: none"> • Reference $\frac{\ddot{T}}{c^2 T} = \frac{X''}{X}$ by its equation label; Press the Enter key. • Context Menu: Right-hand Side • Equate to λ and multiply by X. • Context Menu: Move to Left
<p>(2)</p> $\frac{\ddot{T}}{c^2 T} = \frac{X''}{X} \quad (3)$ <p>left hand side →</p> $\frac{\ddot{T}}{c^2 T}$ <p>(4)</p> $\left(\frac{\ddot{T}}{c^2 T} = \lambda\right) c^2 T$ $\ddot{T} = T c^2 \lambda \quad (5)$ <p>move to left →</p> $\ddot{T} - T c^2 \lambda = 0 \quad (6)$	<p>(2)</p> $\frac{\ddot{T}}{c^2 T} = \frac{X''}{X} \quad (7)$ <p>right hand side →</p> $\frac{X''}{X}$ <p>(8)</p> $\left(\frac{X''}{X} = \lambda\right) X$ $X'' = X \lambda \quad (9)$ <p>move to left →</p> $X'' - X \lambda = 0 \quad (10)$
<p>Table 1 Separating variables in the one-dimensional wave equation</p>	

The boundary conditions $u(0, t) = u(\pi, t) = 0$ yield the homogeneous Dirichlet conditions $X(0) = X(\pi) = 0$, which are separated.

▼ Solving the Sturm-Liouville Eigenvalue Problem

Maple's **dsolve** command (invoked through the Context Menu), cannot solve even a simple a Sturm-Liouville eigenvalue problem:

<ul style="list-style-type: none"> • Context Menu: Solve DE>$X(x)$ 	$X'' - X\lambda = 0, X(0) = 0, X(\pi) = 0 \xrightarrow{\text{solve DE}} X = 0$
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However, with a little adroit intervention, this shortcoming can be overcome.

First, ascertain any value of λ for which the characteristic equation has a multiple root. Find the general solution of that solution separately from the general solution in the case where the characteristic roots are distinct.

Apply the boundary conditions to both of these general solutions. Write the resulting (homogeneous) equations in matrix form, and set the determinant equal to zero. Solve the resulting equation for the eigenvalues, and use these eigenvalues to specify the associated eigenfunctions. The technique is illustrated in the following two examples.

▼ Homogeneous Dirichlet Conditions

Make the "exponential guess" $X = e^{mx}$ to obtain and solve the characteristic equation.

<ul style="list-style-type: none"> • Expression palette: Evaluation template Press the Enter key. • Context Menu: Solve>Obtain Solutions for $>m$ 	$X'' - X\lambda = 0 \quad \left \begin{array}{l} X = e^{mx} \\ m^2 e^{mx} - e^{mx} \lambda = 0 \end{array} \right.$ <p style="text-align: center;">solutions for m \rightarrow</p> $\sqrt{\lambda}, -\sqrt{\lambda}$
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The characteristic equation $m^2 - \lambda = 0$ shows that for $\lambda = 0$, the ODE $X'' = 0$ has the double root $m = 0$, so the general solution in this case is a polynomial.

<ul style="list-style-type: none"> • Expression palette: Evaluation template Press the Enter key. • Context Menu: Solve DE $>X(x)$ • Context Menu: Right-hand Side 	$X'' - X\lambda = 0 \quad \left \begin{array}{l} \lambda = 0 \\ X'' = 0 \end{array} \right. \quad (11)$ <p style="text-align: center;">solve DE \rightarrow</p> $X = _C1 x + _C2 \quad (12)$ <p style="text-align: center;">right hand side \rightarrow</p> $_C1 x + _C2 \quad (13)$
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Aside: Right after obtaining (12), it would seem natural to invoke the Context Menu option "Assign Function", thereby obtaining the solution $X(x)$ as a *function*. Then, the boundary conditions $X(0) = X(\pi) = 0$ would most easily be obtained. Unfortunately, this would render the differential equation (10) opaque to the Context Menu system thereafter. Since (10) is still needed several times in the sequel, the additional step (13) allows the solution to be given the different name, S .

<ul style="list-style-type: none"> • Context Menu: Assign Function 	$S(x) = (13) \xrightarrow{\text{assign as function}} S$
<ul style="list-style-type: none"> • Apply the boundary conditions to $S(x)$; Press the Enter key. • Context Menu: Solve>Solve 	$S(0) = 0, S(\pi) = 0$ $_C2 = 0, _C1 \pi + _C2 = 0$ <p style="text-align: center;">solve \rightarrow</p> $\{ _C1 = 0, _C2 = 0 \}$

Clearly, this implies $X(x) = 0$, so $\lambda = 0$ is not an eigenvalue. The general case where $\lambda \neq 0$ proceeds along similar lines, except for the algebra of solving for the eigenvalues. Begin by unassigning S .

Unassign the name S

• Context Menu: Unassign	$S \xrightarrow{\text{unassign}} S$
Obtain the general solution of the ODE as the function S	
• Reference the ODE by equation label; Press the Enter key.	(10)
• Context Menu: Solve DE	$X'' - X\lambda = 0$ (14)
• Context Menu: Right-hand Side	$\xrightarrow{\text{solve DE}}$
• Write $S(x) = \dots$ (using equation label) Context Menu: Assign Function	$X = _C1 e^{\sqrt{\lambda} x} + _C2 e^{-\sqrt{\lambda} x}$ (15)
	$\xrightarrow{\text{right hand side}}$
	$_C1 e^{\sqrt{\lambda} x} + _C2 e^{-\sqrt{\lambda} x}$ (16)
	$S(x) = \text{(16)} \xrightarrow{\text{assign as function}} S$

At this point, access to the tools of linear algebra becomes necessary.

• Tools>Load Package: Student Linear Algebra	Loading Student:-LinearAlgebra
Apply the boundary conditions and solve for λ	
<ul style="list-style-type: none"> • Apply the boundary conditions and press the Enter key. • Context Menu: Student Linear Algebra>Constructions>Generate Matrix> $_C1, _C2$ • Context Menu: Select Element>1 • Context Menu: Student Linear Algebra>Standard Operations>Determinant • Context Menu: Solve>Solve (general solution) 	
$S(0) = 0, S(\pi) = 0$	
$_C1 + _C2 = 0, _C1 e^{\sqrt{\lambda} \pi} + _C2 e^{-\sqrt{\lambda} \pi} = 0$	
$\xrightarrow{\text{to Matrix form}}$	$\begin{bmatrix} 1 & 1 \\ e^{\sqrt{\lambda} \pi} & e^{-\sqrt{\lambda} \pi} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$\xrightarrow{\text{select entry 1}}$	$\begin{bmatrix} 1 & 1 \\ e^{\sqrt{\lambda} \pi} & e^{-\sqrt{\lambda} \pi} \end{bmatrix}$
$\xrightarrow{\text{determinant}}$	$e^{-\sqrt{\lambda} \pi} - e^{\sqrt{\lambda} \pi}$
$\xrightarrow{\text{solve}}$	$\{\lambda = -Z^2\}$

The symbol $_Z1$ represents an arbitrary integer, which henceforth will be called n . It remains to see what happens to the general solution of the ODE when $\lambda = -n^2$, for integer n . It suffices to take n as positive.

<ul style="list-style-type: none"> • Expression palette: Evaluation Template Press the Enter key. • Context Menu: Simplify>Assuming Positive • Context Menu: Complex Maps>$a + b I$ form • Context Menu: Collect>Function>$\cos(n x)$ 	
$S(x) \Big _{\lambda = -n^2}$	$_C1 e^{\sqrt{-n^2} x} + _C2 e^{-\sqrt{-n^2} x} \quad (17)$
assuming positive \longrightarrow	$_C1 e^{Inx} + _C2 e^{-Inx} \quad (18)$
a + bI form \longrightarrow	$_C1 \cos(n x) + _C2 \cos(n x) + I (_C1 \sin(n x) - _C2 \sin(n x)) \quad (19)$
collect $\underline{\underline{=}}$	$(_C1 + _C2) \cos(n x) + I (_C1 \sin(n x) - _C2 \sin(n x)) \quad (20)$

Is all of (20) the eigenfunction? In fact, does all of (20) even satisfy the boundary conditions?

Apply the boundary conditions to (20)	
<ul style="list-style-type: none"> • Expression palette: Evaluation template Press the Enter key. 	$(20) \Big _{x=0} = 0$ $_C1 + _C2 = 0 \quad (21)$
<ul style="list-style-type: none"> • Expression palette: Evaluation template (Include result from first condition.) Press the Enter key. • Context Menu: Simplify>Simplify 	$(20) \Big _{x=\pi} = 0$ $I (_C1 \sin(n \pi) - _C2 \sin(n \pi)) = 0 \quad (22)$ $\underline{\underline{\text{simplify}}}$ $I \sin(n \pi) (_C1 - _C2) = 0 \quad (23)$

The eigenfunctions are therefore multiples of $\sin(n x)$, $n = 1, \dots$

▼ Homogeneous Neumann Conditions

Subjecting $X = _C1 x + _C2$, the solution of the differential equation $X'' = 0$, to the homogeneous Neumann conditions $X'(0) = X'(\pi) = 0$ results in $_C2 = 0$ and $_C1$ undetermined. Hence, $\lambda = 0$ is now an eigenvalue, with eigenfunction 1.

For the general case $\lambda \neq 0$, the solution $S(x)$ remains a valid starting point.

Apply the Neumann conditions

<ul style="list-style-type: none"> Apply the appropriate condition and press the Enter key. 	
$S'(0) = 0$ $_C1 \sqrt{\lambda} - _C2 \sqrt{\lambda} = 0 \quad (24)$	$S'(\pi) = 0$ $_C1 \sqrt{\lambda} e^{\sqrt{\lambda} \pi} - _C2 \sqrt{\lambda} e^{-\sqrt{\lambda} \pi} = 0 \quad (25)$
Write the equations in Matrix form, set determinant to zero, and solve	
<ul style="list-style-type: none"> Reference equations via equation labels and press the Enter key. Context Menu: Student Linear Algebra>Constructions>Generate Matrix>_C1, _C2 Context Menu: Select Element>1 Context Menu: Student Linear Algebra>Standard Operations>Determinant Context Menu: Solve>Solve (general solution) 	
(24), (25) $_C1 \sqrt{\lambda} - _C2 \sqrt{\lambda} = 0, _C1 \sqrt{\lambda} e^{\sqrt{\lambda} \pi} - _C2 \sqrt{\lambda} e^{-\sqrt{\lambda} \pi} = 0 \quad (26)$	
to Matrix form \longrightarrow	$\begin{bmatrix} \sqrt{\lambda} & -\sqrt{\lambda} \\ \sqrt{\lambda} e^{\sqrt{\lambda} \pi} & -\sqrt{\lambda} e^{-\sqrt{\lambda} \pi} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (27)$
select entry 1 \longrightarrow	$\begin{bmatrix} \sqrt{\lambda} & -\sqrt{\lambda} \\ \sqrt{\lambda} e^{\sqrt{\lambda} \pi} & -\sqrt{\lambda} e^{-\sqrt{\lambda} \pi} \end{bmatrix} \quad (28)$
determinant \longrightarrow	$-\lambda e^{-\sqrt{\lambda} \pi} + \lambda e^{\sqrt{\lambda} \pi} \quad (29)$
solve \longrightarrow	$\{\lambda = 0\}, \{\lambda = -_Z3^2\} \quad (30)$

The eigenvalue $\lambda = 0$ is contained in the general solution, and the nonzero eigenvalues are again $-n^2$, where n is a positive integer. The corresponding eigenfunctions are found next with calculations similar to those employed for the Dirichlet conditions. However, for the Neumann conditions, collect the sine terms, then differentiate with respect to x .

<ul style="list-style-type: none"> Expression palette: Evaluation Template Press the Enter key. Context Menu: Simplify>Assuming Positive Context Menu: Complex Maps>$a + b I$ form Context Menu: Collect>Function>$\sin(n x)$ Context Menu: Differentiate>With Respect To>x

$S(x) \Big _{\lambda = -n^2}$	$_C1 e^{\sqrt{-n^2} x} + _C2 e^{-\sqrt{-n^2} x}$	(31)
assuming positive →	$_C1 e^{Inx} + _C2 e^{-Inx}$	(32)
a + bI form →	$_C1 \cos(nx) + _C2 \cos(nx) + I (_C1 \sin(nx) - _C2 \sin(nx))$	(33)
collect =	$I (_C1 - _C2) \sin(nx) + _C1 \cos(nx) + _C2 \cos(nx)$	(34)
differentiate w.r.t. x →	$I (_C1 - _C2) n \cos(nx) - _C1 n \sin(nx) - _C2 n \sin(nx)$	(35)

Again apply the Neumann conditions to the solution in the form of (34), noting that (35) is its derivative.

Apply the boundary conditions to (20)	
<ul style="list-style-type: none"> • Expression palette: Evaluation template Press the Enter key. 	$(35) \Big _{x=0} = 0$ $I (_C1 - _C2) n = 0$ <p style="text-align: right;">(36)</p>
<ul style="list-style-type: none"> • Expression palette: Evaluation template (Include result from first condition.) Press the Enter key. • Context Menu: Simplify > Assuming Integer 	$(35) \Big _{x=\pi, (36)} = 0$ $I (_C1 - _C2) n \cos(n\pi)$ $- _C1 n \sin(n\pi)$ $- _C2 n \sin(n\pi) = 0$ <p style="text-align: center;">assuming integer →</p> $I (_C1 - _C2) n (-1)^n = 0$ <p style="text-align: right;">(38)</p>

From (38), deduce that the eigenfunctions are multiples of $\cos(nx)$.

▼ Example: An ODE That Is Not Self-Adjoint

Consider the eigenvalue problem defined by the differential equation $y'' + 2y' + \lambda y = 0$, which is not formally self-adjoint, and the homogeneous boundary conditions

$$\begin{aligned} 3y'(0) + 5y(0) &= 0 \\ 2y'(\pi) + y(\pi) &= 0 \end{aligned}$$

The characteristic equation, obtained by making the exponential guess $y = e^{mx}$, is $m^2 + 2m + \lambda = 0$,

with solutions $m = -1 \pm \sqrt{1 - \lambda}$. Thus, $\lambda = 1$ means $m = -1$ is a double root, and for that case, $y = (a + b x) e^{-x}$. To this solution apply the boundary conditions to see if $\lambda = 1$ is an eigenvalue.

$$m = -1 \pm \sqrt{1 - \lambda} \quad (39)$$

The work-flow here is to make the name S available so it can become the name of the solution, then to apply the boundary conditions to $S(x)$, and to examine the resulting homogeneous equations in matrix form. If the determinant of the resulting algebraic system is nonzero, then the only solution will be the null solution, and hence, $y(x) = 0$ and $\lambda = 1$ is not an eigenvalue.

• Context Menu: Unassign	$S \xrightarrow{\text{unassign}} S$
• Context Menu: Assign Function	$S(x) = (a + b x) e^{-x} \xrightarrow{\text{assign as function}} S$
<ul style="list-style-type: none"> • Apply the boundary conditions to the function $S(x)$ and press the Enter key. • Context Menu: Student Linear Algebra > Constructions > Generate Matrix > a, b • Context Menu: Select Element > 1 • Context Menu: Student Linear Algebra > Standard Operations > Determinant 	
$3 S'(0) + 5 S(0) = 0, 2 S'(\pi) + S(\pi) = 0$ $3 b + 2 a = 0, 2 b e^{-\pi} - (b \pi + a) e^{-\pi} = 0$ to Matrix form \rightarrow $\begin{bmatrix} 2 & 3 \\ -e^{-\pi} & -\pi e^{-\pi} + 2 e^{-\pi} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ select entry 1 \rightarrow $\begin{bmatrix} 2 & 3 \\ -e^{-\pi} & -\pi e^{-\pi} + 2 e^{-\pi} \end{bmatrix}$ determinant \rightarrow $-2 \pi e^{-\pi} + 7 e^{-\pi}$	

Next, apply the boundary conditions to the solution of the ODE when $\lambda \neq 1$. The resulting homogeneous equations are cast into matrix form, and the determinant of the system matrix is set equal to zero. The eigenvalues are any values of λ for which this determinant is zero.

• Tools > Load Package: Student Calculus 1	Loading Student:-Calculus1
• Context Menu: Unassign	$S \xrightarrow{\text{unassign}} S$
• Context Menu: Assign Function	$S(x) = a e^{(\sqrt{1-\lambda} - 1)x} + b e^{-(\sqrt{1-\lambda} + 1)x} \xrightarrow{\text{assign as function}} S$

- Apply the boundary conditions to the function $S(x)$ and press the Enter key.
- Context Menu: Student Linear Algebra>Constructions>Generate Matrix> a, b
- Context Menu: Select Element>1
- Context Menu: Student Linear Algebra>Standard Operations>Determinant
- Context Menu: Simplify>Simplify
- Context Menu: Collect>Function>(Select the first exponential expression as per Figure 2.)
- Context Menu: Student Calculus1>Solve>Find Roots>(Complete the dialog as per Figure 3.)
- Context Menu: Assign to a Name>Lambda

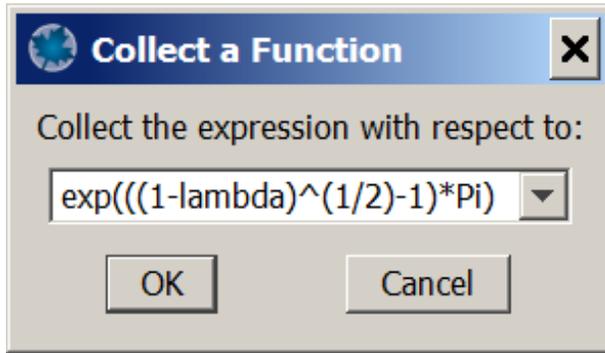


Figure 2 Collect dialog

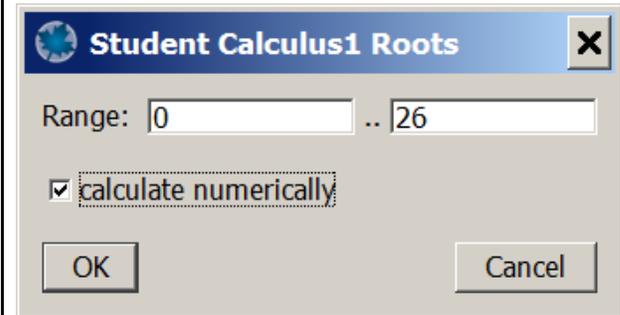


Figure 3 Roots dialog

$$3 S'(0) + 5 S(0) = 0, 2 S'(\pi) + S(\pi) = 0$$

$$3 a (\sqrt{1-\lambda} - 1) + 3 b (-\sqrt{1-\lambda} - 1) + 5 a + 5 b = 0, 2 a (\sqrt{1-\lambda} - 1) e^{(\sqrt{1-\lambda} - 1)\pi} + 2 b (-\sqrt{1-\lambda} - 1) e^{-(\sqrt{1-\lambda} + 1)\pi} + a e^{(\sqrt{1-\lambda} - 1)\pi} + b e^{-(\sqrt{1-\lambda} + 1)\pi} = 0$$

to Matrix form

$$\left[\begin{bmatrix} 3\sqrt{1-\lambda} + 2, & -3\sqrt{1-\lambda} + 2 \\ 2(\sqrt{1-\lambda} - 1) e^{(\sqrt{1-\lambda} - 1)\pi} + e^{(\sqrt{1-\lambda} - 1)\pi}, & 2(-\sqrt{1-\lambda} - 1) e^{-(\sqrt{1-\lambda} + 1)\pi} + e^{-(\sqrt{1-\lambda} + 1)\pi} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right]$$

select entry 1

$$\left[\begin{bmatrix} 3\sqrt{1-\lambda} + 2, & -3\sqrt{1-\lambda} + 2 \\ 2(\sqrt{1-\lambda} - 1) e^{(\sqrt{1-\lambda} - 1)\pi} + e^{(\sqrt{1-\lambda} - 1)\pi}, & 2(-\sqrt{1-\lambda} - 1) e^{-(\sqrt{1-\lambda} + 1)\pi} + e^{-(\sqrt{1-\lambda} + 1)\pi} \end{bmatrix}, \right]$$

determinant

$$-7 e^{(\sqrt{1-\lambda} - 1)\pi} \sqrt{1-\lambda} - 7 e^{-(\sqrt{1-\lambda} + 1)\pi} \sqrt{1-\lambda} - 6 e^{(\sqrt{1-\lambda} - 1)\pi} \lambda + 6 e^{-(\sqrt{1-\lambda} + 1)\pi} \lambda + 8 e^{(\sqrt{1-\lambda} - 1)\pi} - 8 e^{-(\sqrt{1-\lambda} + 1)\pi}$$

simplify

$$(6\lambda - 7\sqrt{1-\lambda} - 8) e^{-(\sqrt{1-\lambda} + 1)\pi} - 6 \left(\lambda + \frac{7\sqrt{1-\lambda}}{6} - \frac{4}{3} \right) e^{(\sqrt{1-\lambda} - 1)\pi}$$

collect

$$(-6\lambda - 7\sqrt{1-\lambda} + 8) e^{(\sqrt{1-\lambda} - 1)\pi} + (6\lambda - 7\sqrt{1-\lambda} - 8) e^{-(\sqrt{1-\lambda} + 1)\pi}$$

roots

$$[0.4558571181, 1.116356583, 4.243407309, 9.251574987, 16.25414949, 25.25529797]$$

assign to a name

Λ

The six eigenvalues in the list Λ can only be obtained numerically. Note that the first eigenvalue is less than 1, whereas the remaining ones are greater than 1. This means that in the expression for the eigenfunctions, $\sqrt{1-\lambda}$ will be real for the first eigenvalue and complex for all the others. This has implications for the corresponding eigenfunctions.

To find these eigenfunctions, first solve the first boundary condition for a , and replace a in the general solution. Since b is then an arbitrary multiple, set it to 1. The resulting form of the solution will yield the first eigenfunction corresponding to the eigenvalue that is less than 1.

- Apply the first boundary condition to $S(x)$ and press the Enter key.
- Context Menu: Solve>Isolate Expression for> a
- Context Menu: Substitute Into> $S(x)$
- Context Menu: Evaluate at a Point> $b = 1$

$$3 S'(0) + 5 S(0) = 0$$

$$3 a (\sqrt{1-\lambda} - 1) + 3 b (-\sqrt{1-\lambda} - 1) + 5 a + 5 b = 0 \quad (40)$$

isolate for a
→

$$a = \frac{-3 b (-\sqrt{1-\lambda} - 1) - 5 b}{3 \sqrt{1-\lambda} + 2} \quad (41)$$

substitute into
→

$$\frac{(-3 b (-\sqrt{1-\lambda} - 1) - 5 b) e^{(\sqrt{1-\lambda} - 1)x}}{3 \sqrt{1-\lambda} + 2} + b e^{-(\sqrt{1-\lambda} + 1)x} \quad (42)$$

evaluate at point
→

$$\frac{(3 \sqrt{1-\lambda} - 2) e^{(\sqrt{1-\lambda} - 1)x}}{3 \sqrt{1-\lambda} + 2} + e^{-(\sqrt{1-\lambda} + 1)x} \quad (43)$$

For $\lambda > 1$, (43) is complex. Both the real and imaginary parts are solutions of the ODE, but these two solutions can't be independent - they must be proportional. Hence, select, say, the real part via

$SR := \text{simplify}(\text{evalc}(\text{Re}((43))))$ assuming $\lambda > 1$

$$SR := \frac{18 e^{-x} (-1 + \lambda) \cos(\sqrt{-1 + \lambda} x) - 12 \sqrt{-1 + \lambda} e^{-x} \sin(\sqrt{-1 + \lambda} x)}{-5 + 9 \lambda}$$

Note the switch to explicit Maple commands. While SR could have been obtained with the Context Menu, the *collect* option used next provides for the *simultaneous* collection of terms, a feature not available in a syntax-free form.

Divide SR by its value at $x = 0$ and apply the **collect** command to obtain a still simpler form.

$\Phi := \text{collect}(SR / \text{eval}(SR, x = 0), [\text{exp}, \text{cos}, \text{sin}])$

$$\Phi := \left(\cos(\sqrt{-1 + \lambda} x) - \frac{12 \sqrt{-1 + \lambda} \sin(\sqrt{-1 + \lambda} x)}{-18 + 18 \lambda} \right) e^{-x}$$

Obtain the first six eigenfunctions. (Note that there is no syntax-free way to implement a loop.)

$$\phi_1 := \text{eval}((43), \lambda = \Lambda_1)$$

$$\phi_1 := 0.05055356794 e^{-0.2623395890x} + e^{-1.737660411x}$$

for k **from** 2 **to** 6 **do**

$$\phi_k := \text{eval}(\Phi, \lambda = \Lambda_k);$$

end do;

$$\phi_2 := (\cos(0.3411108075x) - 1.954399138 \sin(0.3411108075x)) e^{-x}$$

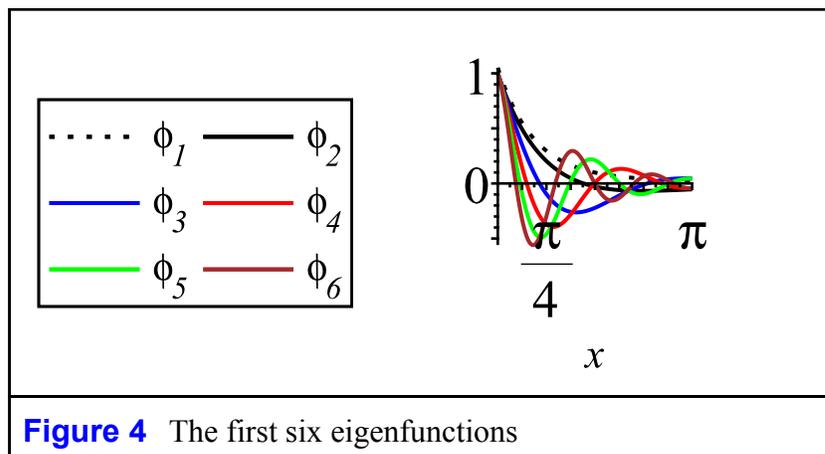
$$\phi_3 := (\cos(1.800946226x) - 0.3701757760 \sin(1.800946226x)) e^{-x}$$

$$\phi_4 := (\cos(2.872555480x) - 0.2320813892 \sin(2.872555480x)) e^{-x}$$

$$\phi_5 := (\cos(3.905656090x) - 0.1706926190 \sin(3.905656090x)) e^{-x}$$

$$\phi_6 := (\cos(4.924966799x) - 0.1353647027 \sin(4.924966799x)) e^{-x}$$

Construct Figure 4, a graph of these six eigenfunctions.



▼ A Numeric Approach

Surprisingly, Maple's **dsolve** command is capable of computing eigenpairs numerically.

Initialize	
• Context Menu: Assign to a Name>DE	$y'' + 2y' + \lambda y = 0 \xrightarrow{\text{assign to a name}} DE$
• Context Menu: Assign to a Name>BC ₁	$3y'(0) + 5y(0) = 0 \xrightarrow{\text{assign to a name}} BC_1$
• Context Menu: Assign to a Name>BC ₂	$2y'(\pi) + y(\pi) = 0 \xrightarrow{\text{assign to a name}} BC_2$
• Context Menu: Assign to a Name>sys	$\{DE, BC_1, BC_2, y(0) = 1\} \xrightarrow{\text{assign to a name}} sys$

Table 2 shows how to obtain eigenpairs numerically. The left-hand column shows how to apply the `dsolve` command; the right, how to extract the computed eigenvalue. In order for λ to be an unknown in these calculations, an additional constraint must be provided. This is the normalization $y(0) = 1$. In addition, good approximations of both the shape of the eigenfunction and the eigenvalue must be provided. This latter is the chief obstacle to this method becoming a practical tool. These approximations become harder and harder to determine. In effect, one needs to have the solution in order to calculate the solution.

$Q_1 := dsolve(sys, numeric) :$	$\lambda \Big _{Q_1(0)} =$ 0.455857086006263
$Q_2 := dsolve(sys, numeric, approxsoln = [y(x) = 1 - x/3, \lambda = 1]) :$	$\lambda \Big _{Q_2(0)} = 1.11635653865587$
$Q_3 := dsolve(sys, numeric, approxsoln = [y(x) = (x - 3/2)^2, \lambda = 4]) :$	$\lambda \Big _{Q_3(0)} = 4.24340701384004$
Table 2 Numeric solution for eigenpairs	

Figure 5 contains graphs of the three eigenfunctions determined numerically.

