

Classroom Tips and Techniques: The Lagrange Multiplier Method

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▼ Introduction

The typical multivariate calculus course contains at least one lesson detailing constrained optimization via the *Lagrange multiplier* method. Once such a problem has been formulated, the hardest part in the implementation of the Lagrange multiplier method is solving the nonlinear equations that result.

There is no recipe by means of which such a system of nonlinear equations can be automatically solved. Classroom lessons on the Lagrange multiplier method usually consist of a list of examples, and once each problem has been formulated, the time-sink is algebra of solving the nonlinear equations the method generates. The more examples worked, the more varied the spectrum of nonlinear systems met. Since there is no algorithm the students can be taught, it is hoped that by demonstrating the solution of enough systems that the students absorb some of the art required.

Throughout, the only advice a student can be given is "do not divide by zero." Thus, every system presents its own challenges, and the only general rule that can be articulated is "during all relevant manipulations, avoid dividing by a quantity that could itself be zero."

In courses where instructors are willing to consign to Maple the task of solving sets of nonlinear equations, the Lagrange multiplier method can be taught and mastered efficiently. My experience at Rose-Hulman Institute of Technology was that with five carefully chosen examples, I could explain the mechanics and theory of the Lagrange multiplier method in one fifty minute class, and give the students insight into the principal difficulties such optimization problems might present.

The first example showed that constrained optima occurred at points where the graph of the constraint was tangent to a level set of the objective function. And that the Lagrange multiplier arose from the search for such points of tangency. In essence, this is all the theory needed to explain and understand the method.

The second example explored the meaning of a multiplier being zero at an extremum. The third example required formulating the objective function for minimizing the distance from a point to a plane. The fourth example contained a parameter, so solving the nonlinear system required a bit more care. The fifth example contained two constraints.

In this article, I will clarify what tools Maple makes available for teaching, learning, and implementing the Lagrange multiplier method at a level consistent with a first course in multivariate calculus.

▼ Example

Obtain all extrema for the objective function $f(x, y) = 2x^2 + 3y^2$ subject to the constraint $g(x, y) \equiv x^2 + y^2 - 1 = 0$.

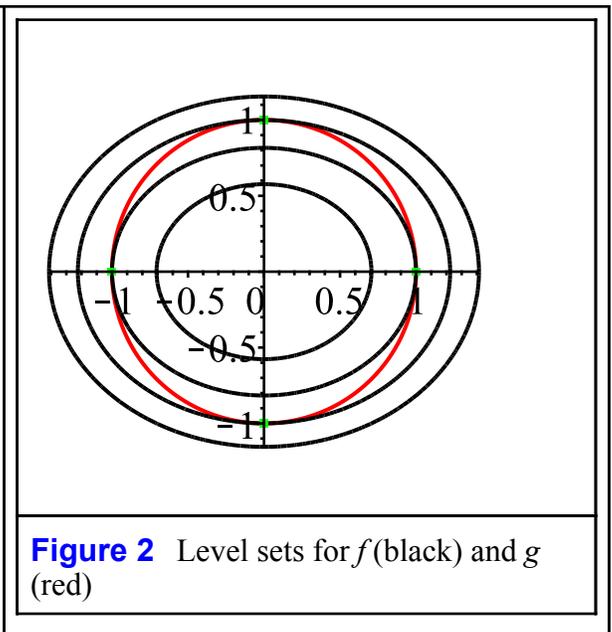
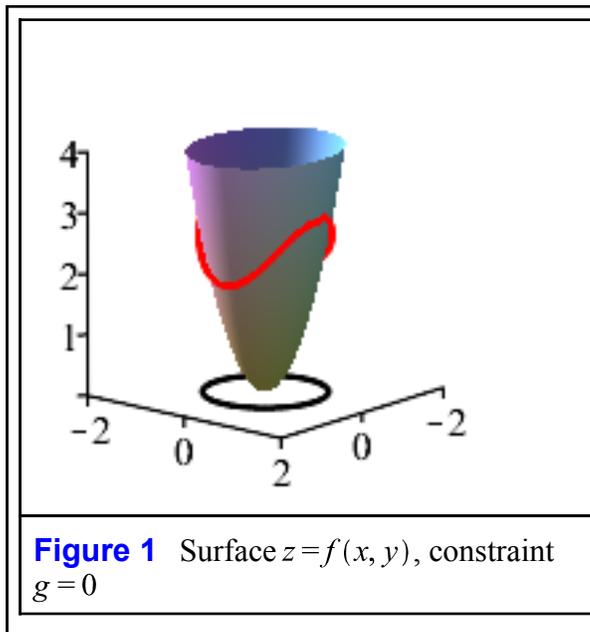
▼ Solution

▼ *Mathematical Solution*

Figure 1 contains a graph of the surface $z = f(x, y)$ and the constraint circle $g = 0$ (in black). The "lift" of the constraint circle onto the surface of the objective function is drawn in red. This figure suggests that there will be two local maxima and two local minima of f along the constraint circle.

Figure 2 shows (in black) the level sets (curves) for the objective function f and (in red) the graph of the constraint circle. The green dots represent the four points where the circle appears to be tangent to a level curve.

These points of tangency are the extrema. This can be seen by the following observations. Traverse the constraint circle in a counterclockwise direction, starting at the point $[1, 0]$. Along this path, level curves for the objective function will be crossed. Since this function increases as the level curves move outward from the origin, function values met along the constraint will increase. These values continue to increase until the point $[0, 1]$ is reached, after which the values of the objective function will begin to decrease. Thus, the point of tangency is an extreme value.

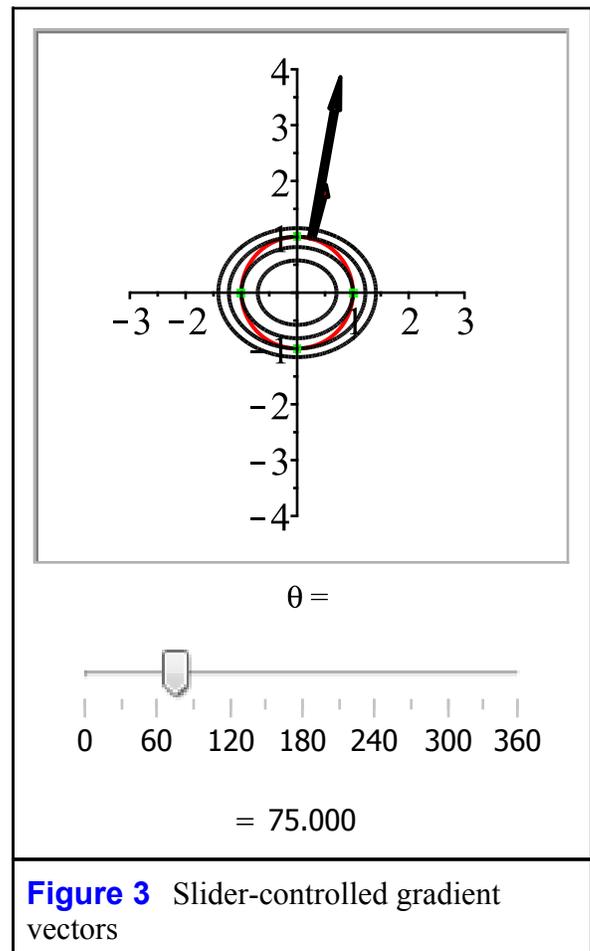


The Lagrange multiplier method seeks to find points of tangency between the level sets of the objective function and the constraint. At such points of tangency, the gradients of both

functions become collinear, a condition expressed by the equation $\nabla f = \lambda \nabla g$. The gradient vectors do not become equal, but merely proportional, and this constant of proportionality is often designated by λ .

For this example, the equation expressing collinearity of the gradient vectors contains just two equations, but the coordinates of an extremum are two unknowns, and λ is a third unknown. The third equation that is needed is the constraint itself. Thus, the Lagrange multiplier method applied to this example requires solving three nonlinear equations in three unknowns.

The slider under the graph in Figure 3 controls the gradient vectors along the constraint circle, the black vector being orthogonal to the level curves of f , the red, to g . Note that these vectors appear to be collinear at each of the four green dots, the points of tangency of the constraint circle and a level curve for the objective function.



For this example, the calculations needed for the implementation of the Lagrange multiplier method consist in solving the equations

$$\nabla f = \begin{bmatrix} 4x \\ 6y \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 2y \end{bmatrix} \text{ and } g = x^2 + y^2 - 1 = 0$$

or

$$4x = 2\lambda x, 6y = 2\lambda y, x^2 + y^2 = 1$$

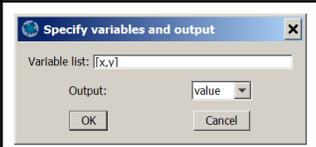
There is no "recipe" for solving nonlinear equations, only one caution: never cancel. Equations can be factored, and the zero-principle applied, but variables should never be cancelled because such a cancellation could represent a division by zero.

Thus, the first two equations can be written as $x(2 - \lambda) = 0$ and $y(3 - \lambda) = 0$, from which it can be determined that if $\lambda = 2$, then $y = 0$, so that $x = \pm 1$; and if $\lambda = 3$, then $x = 0$ so that $y = \pm 1$. Hence, there are four extrema, namely, $(1, 0)$, $(-1, 0)$, $(0, 1)$, $(0, -1)$. Whether an extremum is a local maximum or minimum is generally determined by evaluating the objective function at each such extreme value.

$$f(1, 0) = 2 = f(-1, 0) \text{ and } f(0, 1) = 3 = f(0, -1)$$

Hence, there are two local minima with value 2 and two local maxima with value 3.

▼ Maple Solution - Context Menu

Initialize	
<ul style="list-style-type: none"> Tools > Load Package: Student Multivariate Calculus 	Loading Student:-MultivariateCalculus
Apply the Lagrange multiplier option	
<ul style="list-style-type: none"> Write a sequence of the rules for the <u>functions</u> f and g. Context Menu: Student Multivariate Calculus > Lagrange Multipliers Complete the "Specify variables and output" dialog as per the figure on the right.	
$2x^2 + 3y^2, x^2 + y^2 - 1 \xrightarrow{\text{Lagrange multipliers}} [0, 1], [0, -1], [1, 0], [-1, 0]$	
<ul style="list-style-type: none"> Write a sequence of the rules for the <u>functions</u> f and g. Context Menu: Student Multivariate Calculus > Lagrange Multipliers Select "detailed" for the Output.	
$2x^2 + 3y^2, x^2 + y^2 - 1 \xrightarrow{\text{Lagrange multipliers}}$ $[x=0, y=1, \lambda_1=3, 2x^2 + 3y^2=3], [x=0, y=-1, \lambda_1=3, 2x^2 + 3y^2=3], [x=1, y=0, \lambda_1=2, 2x^2 + 3y^2=2], [x=-1, y=0, \lambda_1=2, 2x^2 + 3y^2=2]$	

The "detailed" option returns the extrema, the values of the Lagrange multiplier for each extremum, and the value of the objective function at each extremum.

Assuming that the Student MultivariateCalculus package has already been installed, the [LagrangeMultipliers](#) command can be accessed directly with the syntax shown below. The output options are "value", "detailed", and "plot", with the default being "value".

$\text{LagrangeMultipliers}(2x^2 + 3y^2, [x^2 + y^2 - 1], [x, y], \text{output} = \text{detailed})$ $[x=0, y=1, \lambda_1=3, 2x^2 + 3y^2=3], [x=0, y=-1, \lambda_1=3, 2x^2 + 3y^2=3], [x=1, y=0, \lambda_1=2, 2x^2 + 3y^2=2], [x=-1, y=0, \lambda_1=2, 2x^2 + 3y^2=2]$
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▼ Maple Solution - First Principles

Instead of forming and solving the equations $\nabla f = \lambda \nabla g, g = 0$, define $F \equiv f - \lambda g$, then form and solve the equations $F_x = F_y = F_\lambda = 0$.

<ul style="list-style-type: none"> • Define F. • Context Menu: Assign Name 	$F = 2x^2 + 3y^2 - \lambda(x^2 + y^2 - 1) \xrightarrow{\text{assign}}$
<ul style="list-style-type: none"> • Calculus palette: Partial derivative operator Press the Enter key. • Context Menu: Solve>Solve 	
$\frac{\partial}{\partial x} F = 0, \frac{\partial}{\partial y} F = 0, \frac{\partial}{\partial \lambda} F = 0$ $-2\lambda x + 4x = 0, -2\lambda y + 6y = 0, -x^2 - y^2 + 1 = 0$ <p style="text-align: center;">$\xrightarrow{\text{solve}}$</p> $\{\lambda = 3, x = 0, y = 1\}, \{\lambda = 3, x = 0, y = -1\}, \{\lambda = 2, x = 1, y = 0\}, \{\lambda = 2, x = -1, y = 0\}$	

All that remains is to evaluate f at the extrema.

The astute reader will realize that the three equations so formed are the equivalent of setting the components of the "gradient" of F to zero. The essential reason for eschewing the equation $\nabla f = \lambda \nabla g$ is the inability to use the Nabla (∇) interactively without installing the Student *VectorCalculus* package. However, it is possible to access the Gradient option in the Context Menu under the aegis of the Student *MultivariateCalculus* package. This requires a lot more referencing and the use of **Equate**, as can be seen in the more tedious solution below.

<ul style="list-style-type: none"> • Write the rule for f (then repeat for g). • Context Menu: Student Multivariate Calculus>Differentiate>Gradient • Context Menu: Assign to a Name>gf (then gg) 	
$2x^2 + 3y^2 \xrightarrow{\text{gradient}} \begin{bmatrix} 4x \\ 6y \end{bmatrix} \xrightarrow{\text{assign to a name}} gf$	$x^2 + y^2 - 1 \xrightarrow{\text{gradient}} \begin{bmatrix} 2x \\ 2y \end{bmatrix} \xrightarrow{\text{assign to a name}} gg$
<ul style="list-style-type: none"> • Form a sequence of ∇f and $\lambda \nabla g$ and press the Enter key. • Context Menu: Equate • Using the equation label, form a sequence of the list of equations in $\nabla f = \lambda \nabla g$ and the list [$g = 0$]. Press the Enter key. • Context Menu: Join • Context Menu: Solve>Solve 	

To obtain other extrema, modify the Initial Values, that is, the starting point for the numeric search that this tool utilizes. (Press the Edit button to the right of "Initial Values" and enter values in place of the words "default".

To obtain the graph shown in Figure 6, check the "as Surfaces" box at the bottom of the Optimization Plotter. This form of the graph shows how the constraint is "lifted" to the surface representing the objective function.

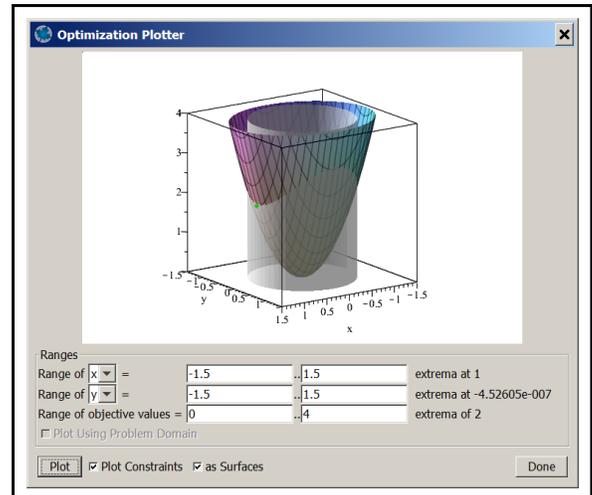


Figure 6 Constraint as a surface

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