pascal1.mws

Permutation of Vertices of Pascal's Hexagon

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Pascal's theorem (1640) says:

If a hexagon H_1 =(123456) is inscribed in a conic, then opposite sides intersect in three points

P_1 =(12.45), Q_1 = (23.56), R_1 =(34.61) which are collinear.

The representation of the hexagon H_1 and intersection points of opposite sides its by symbol is following:

$$H_1 := \begin{bmatrix} 12 & 23 & 34 \\ 45 & 56 & 61 \\ P_1 & Q_1 & R_1 \end{bmatrix}$$

By the way of elementar combinatorics can be proof that the number of different hexagons H_i is 60, and the number of different points P_i , Q_i , R_i the intersections opposite sides of all hexagons are 45.

Let's show this special projective configuration by Maple.!

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> restart; with(combstruct); with(combinat); with(geometry); with(plots)
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Warning, new definition for Chi

Warning, new definition for draw

> $U := \mathrm{permute}([1, 2, 3, 4, 5]);\ h := \mathrm{numbperm}(5);$ **for** i **to** h **do** $bb_i := \mathrm{seq}(U_{i_p}, p = 1 .. 5);\ cc_i := [6, bb_i, 6]$ **end do**

> $\mathrm{circle}(C, x^2 + y^2 = 1, [x, y], centername = \ );$ **for** i **to** 6 **do** $\mathrm{randpoint}(P . i, C)$ **end do**

Let's make coordinates 6 randpoints P.i of circle C and the lines l.i through of points recommanded sequence cc_{z_i} and $cc_{z_{i+1}}$. cc_{z_i} and $cc_{z_{i+1}}$ are element of by permutations generated points.

For example one of hexagons (z = 11) and its sides

> $z := 11;\ cc_z;$ **for** i **to** 6 **do** $\mathrm{line}(l . i, [P . (cc_{z_i}), P . (cc_{z_{i+1}})]);\ \mathrm{print}(l . i, \mathit{line\ through\ of\ points}, cc_{z_i}, \mathit{and}, cc_{z_{i+1}})$ **end do**

$$[6, 1, 3, 5, 2, 4, 6]$$

$$l1, \mathit{line\ through\ of\ points}, 6, \mathit{and}, 1$$

$$l2, \mathit{line\ through\ of\ points}, 1, \mathit{and}, 3$$

$$l3, \mathit{line\ through\ of\ points}, 3, \mathit{and}, 5$$

$$l4, \mathit{line\ through\ of\ points}, 5, \mathit{and}, 2$$

$$l5, \mathit{line\ through\ of\ points}, 2, \mathit{and}, 4$$

$$l6, \mathit{line\ through\ of\ points}, 4, \mathit{and}, 6$$

The intersections of opposite sides l.1, l.4 is P, l.2, l.5 is Q, l.3, l.6 is S.

Make the line T_1 through of the points P and S, T_2 through of the points Q and S, T_3 through of the points P and Q,

and show the Pascal's hexagon and line.

> $\mathrm{intersection}(P, l . 1, l . 4);$

$intersection(Q, l.2, l.5);$
$intersection(S, l.3, l.6);$
$line(T\_1, [P, S], [x, y]);$
$line(T\_2, [Q, S], [x, y]);$
$line(T\_3, [Q, P], [x, y])$

The line T_1 through of points P and S, the line T_2 through of points Q and S, or the line T_3 through Q and P are equal by accuracy of calculating method - look the original Pascal's theorem.

> $Y.1 := solve(Equation(T\_1), y); Y.2 := solve(Equation(T\_2), y); Y.3 := solve(Equation(T\_3), y)$

$$Y1 := -.3605438025 - 1.513607195\, x$$

$$Y2 := -.3605437997 - 1.513607193\, x$$

$$Y3 := -.3605438019 - 1.513607196\, x$$

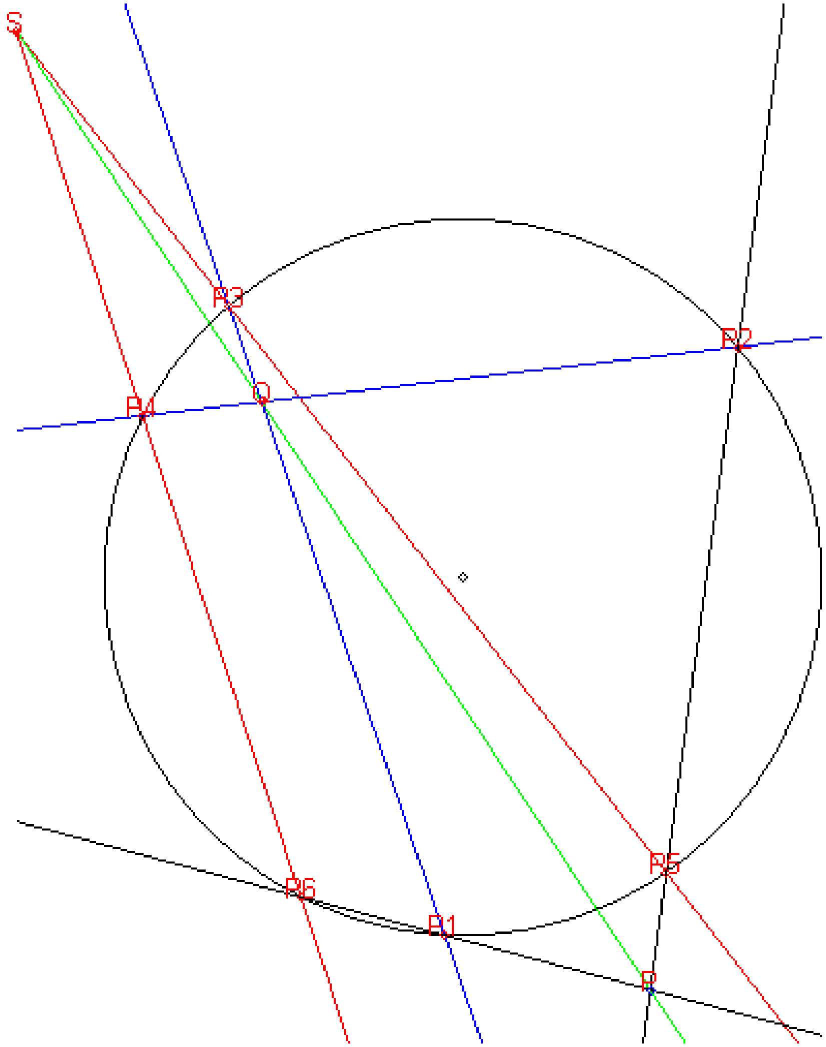
> $(Y.1) - (Y.2); (Y.1) - (Y.3); (Y.2) - (Y.3)$

$$-.28\ 10^{-8} - .2\ 10^{-8}\, x$$

$$-.6\ 10^{-9} + .1\ 10^{-8}\, x$$

$$.22\ 10^{-8} + .3\ 10^{-8}\, x$$

> $pic := draw(\{C(color = black), P.1, P.2, P.3, P.4, P.5, P.6, l.(1(color = black)), l.(4(color = black)), l.(2(color = blue)),$
$l.(5(color = blue)), l.3, l.6, P(color = blue), Q, S, T\_1(color = green)\}, scaling = constrained, axes = none, printtext = true);$
$display(pic, view = [-1.3 .. 1.3, -1.3 .. 1.6], axes = none)$



Finally make the all hexagons, points and lines of combinatorical structure cc[i]. By simple way of representation show the full combinatorical and geometrical structure.

> **for** q **to** h **do**

for i **to** 6 **do** line$(l\,.\,i, [P\,.\,(cc_{q_i}), P\,.\,(cc_{q_{i+1}})])$ **end do**;

intersection$(P, l\,.\,1, l\,.\,4)$;

intersection$(Q, l\,.\,2, l\,.\,5)$;

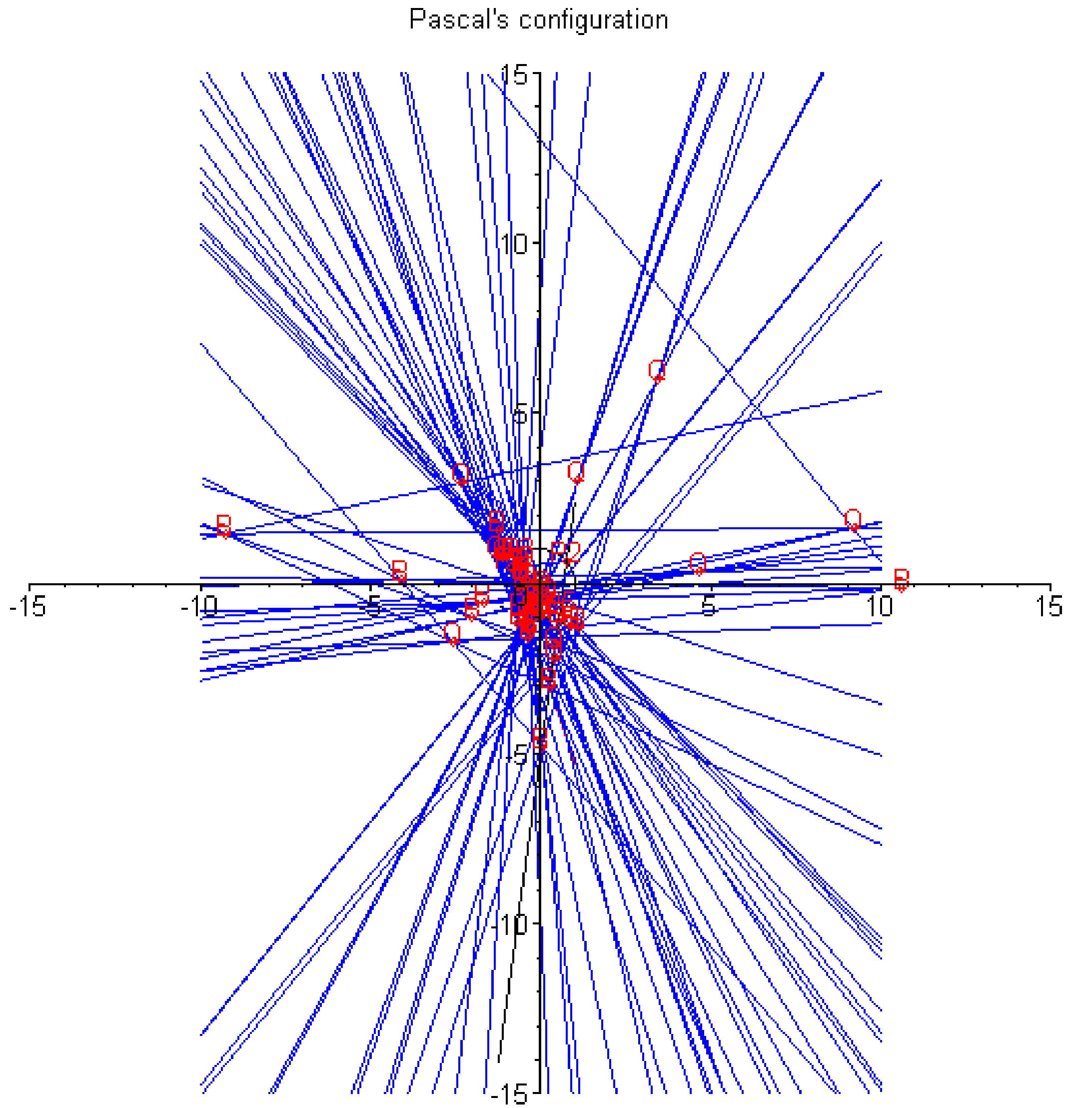
intersection$(S, l\,.\,3, l\,.\,6)$;

line$(T, [P, Q])$;

$kep_q := \mathrm{draw}(\{P, Q, S, \mathrm{T}(color = blue)\}, scaling = constrained, axes = none, printtext = true)$

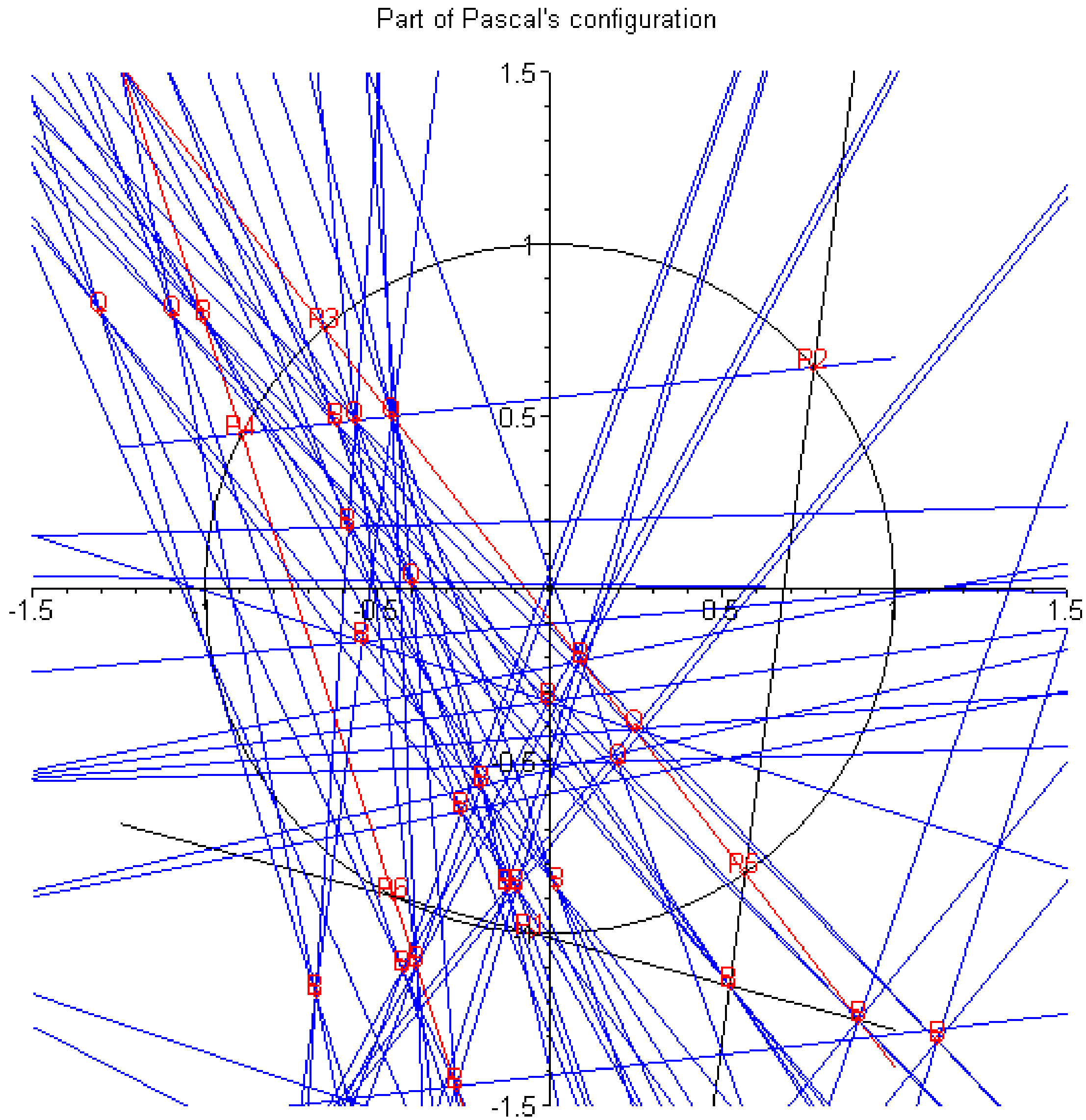
end do

> $\mathrm{display}(\{\mathrm{seq}(kep_w, w = 1 .. h)\}, pic, title = Pascal's\ configuration, view = [-15 .. 15, -15 .. 15], axes = normal, scaling = constrained)$



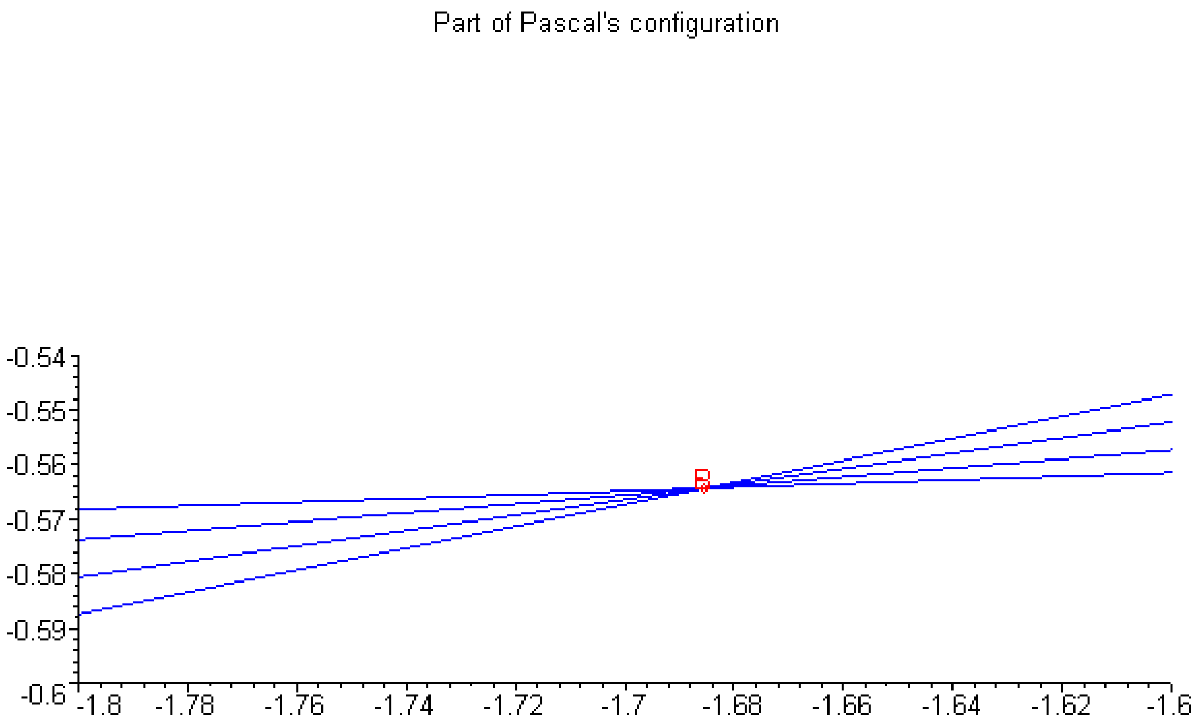
>

$\mathrm{display}(\{\mathrm{seq}(kep_w, w = 1 .. h)\}, pic, title = Part\ of\ Pascal's\ configuration, view = [-1.5 .. 1.5, -1.5 .. 1.5], axes = normal, scaling = constrained)$



>

display({seq(kep_w, w = 1 .. h)}, *pic*, *title* = *Part of Pascal's configuration*, *view* = [−1.8 .. −1.6, −.54 .. −.6], *axes* = *normal*, *scaling* = *constrained*)


Part of Pascal's configuration
-0.54
-0.55
-0.56
-0.57
-0.58
-0.59
-0.6
-1.8
-1.78
-1.76
-1.74
-1.72
-1.7
-1.68
-1.66
-1.64
-1.62
-1.6
P

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