Application Demonstration

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The orthogonal series expansions package for Maple

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Introduction

The **OrthogonalExpansions** package is a collection of commands to compute one-dimensional and multi-dimensional orthogonal series expansions of a function. The expansions are evaluated symbolically, numerically or in an inert form. In particular, the package includes six modifications of the one-dimensional Fourier series, as well the multi-dimensional Fourier, cosine and sine series, one- and multi-dimensional series expansions of classical orthogonal polynomials.

The package includes **MixedSeries** command to compute multi-dimensional mixed orthogonal series expansions. Mixed series are created from the one-dimensional and multi-dimensional series that are available in the package.

The package also includes **GramSchmidtL2** command for computing symbolically or numerically an orthonormal set of one- and multi-variable functions.

Package installation instruction

```
> restart :
> with(OrthogonalExpansions);
```

[BesselSeries, ChebyshevTSeries, ChebyshevUSeries, FourierSeries, GegenbauerSeries, GramSchmidtL2, Haar, HaarSeries, HarmonicWavelet, HarmonicWaveletSeries, HermiteSeries, JacobiSeries, LaguerreSeries, LegendreSeries, MixedSeries, Rational, RationalSeries, RectSeries, SincSeries, SincWavelet, SincWaveletSeries, SphericalSeries, Walsh, WalshSeries, Zernike, ZernikeSeries] **(1)**

```
> ?OrthogonalExpansions;
```

Some orthogonal functions

The package includes a few orthonormal function sets: Haar wavelets, Harmonic wavelets, Rational orthogonal functions, Sinc wavelets, Walsh functions. We briefly consider some of them.

Haar wavelets

The **Haar(n, j, x)** calling sequences computes the Haar wavelet with scale **n** and shift **j**. These wavelets are orthonormal on the interval **[0, 1]**. They satisfy the following:

$$\int_0^1 Haar(n, j, x)\, Haar(m, k, x)\, \mathrm{d}x = \delta_{nm} \cdot \delta_{jk}.$$

> $Haar(n, j, x);$

$$2^{\frac{1}{2} n} \left(\begin{cases} 1 & \frac{2j}{2^{n+1}} \le x \textbf{ and } x < \frac{2j+1}{2^{n+1}} \\ -1 & \frac{2j+1}{2^{n+1}} \le x \textbf{ and } x < \frac{2j+2}{2^{n+1}} \\ 0 & otherwise \end{cases} \right) \quad \textbf{(3.1.1)}$$

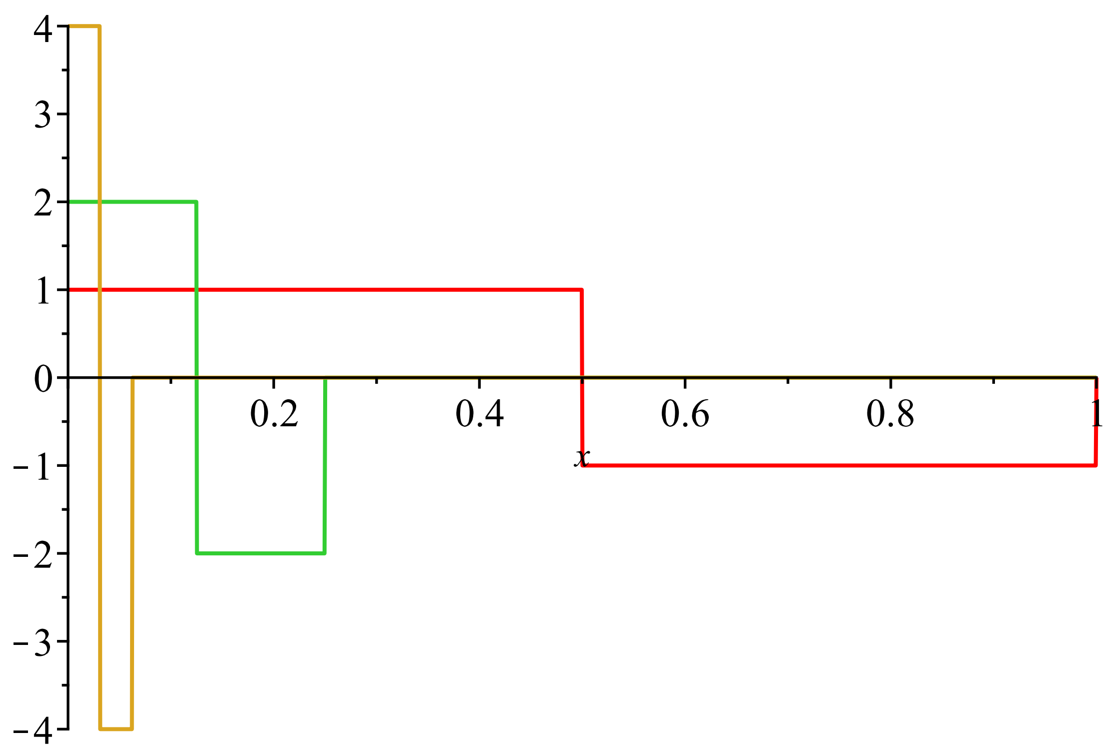
Orthogonality checking

> $\int_0^1 Haar(0, 0, x)\, Haar(2, 0, x)\, \mathrm{d}x, \int_0^1 Haar(2, 0, x)\, Haar(2, 1, x)\, \mathrm{d}x;$

$$0, 0 \quad \textbf{(3.1.2)}$$

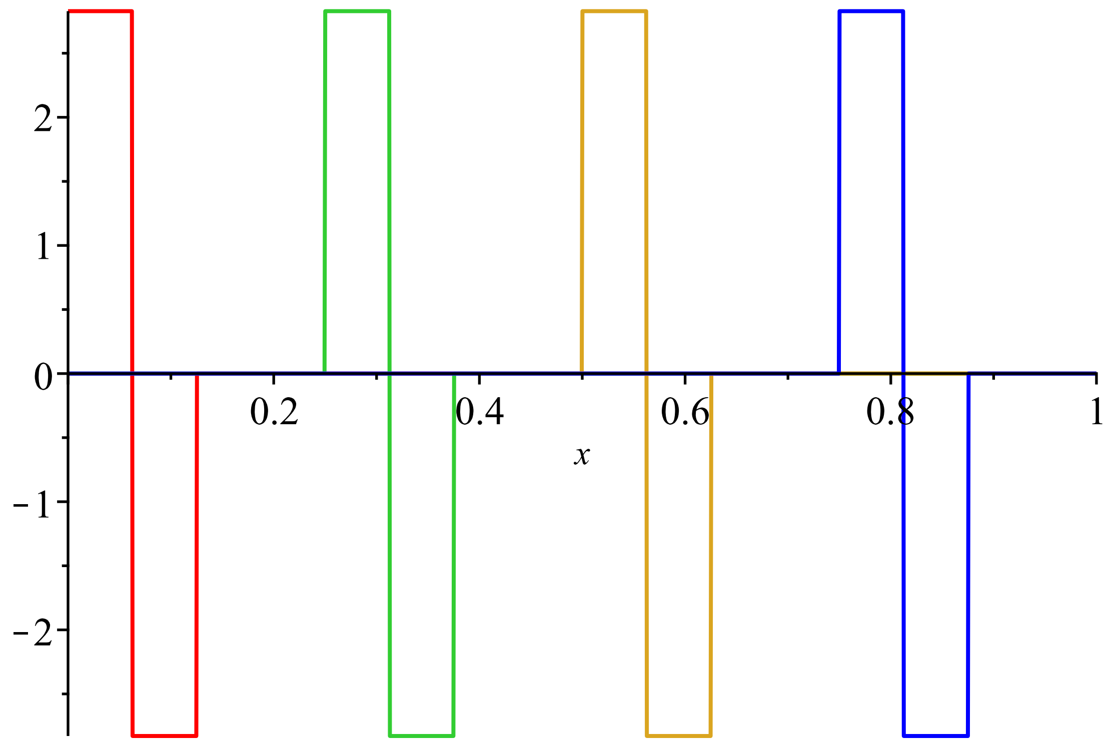
Haar wavelets with different scales

> $plot([Haar(0, 0, x), Haar(2, 0, x), Haar(4, 0, x)], x = 0..1);$



Haar wavelets with different shifts

> $plot([Haar(3, 0, x), Haar(3, 2, x), Haar(3, 4, x), Haar(3, 6, x)], x = 0..1);$



>

Rational orthogonal functions

This functions are good alternative to Hermite polynomials for series expansion on infinite interval. The Rational(j, x) calling sequences computes the Orthogonal Rational function with number j. These functions are orthonormal on the interval (-infinity, infinity). They satisfy the following:

$$\int_{-\infty}^{\infty} Rational(j, x)\, \overline{Rational(k, x)}\; dx = \delta_{jk}.$$

> $Rational(j, x);$

$$\frac{(1 + \mathrm{I}\, x)^{j}}{\sqrt{\pi}\, (1 - \mathrm{I}\, x)^{j+1}} \qquad \textbf{(3.2.1)}$$

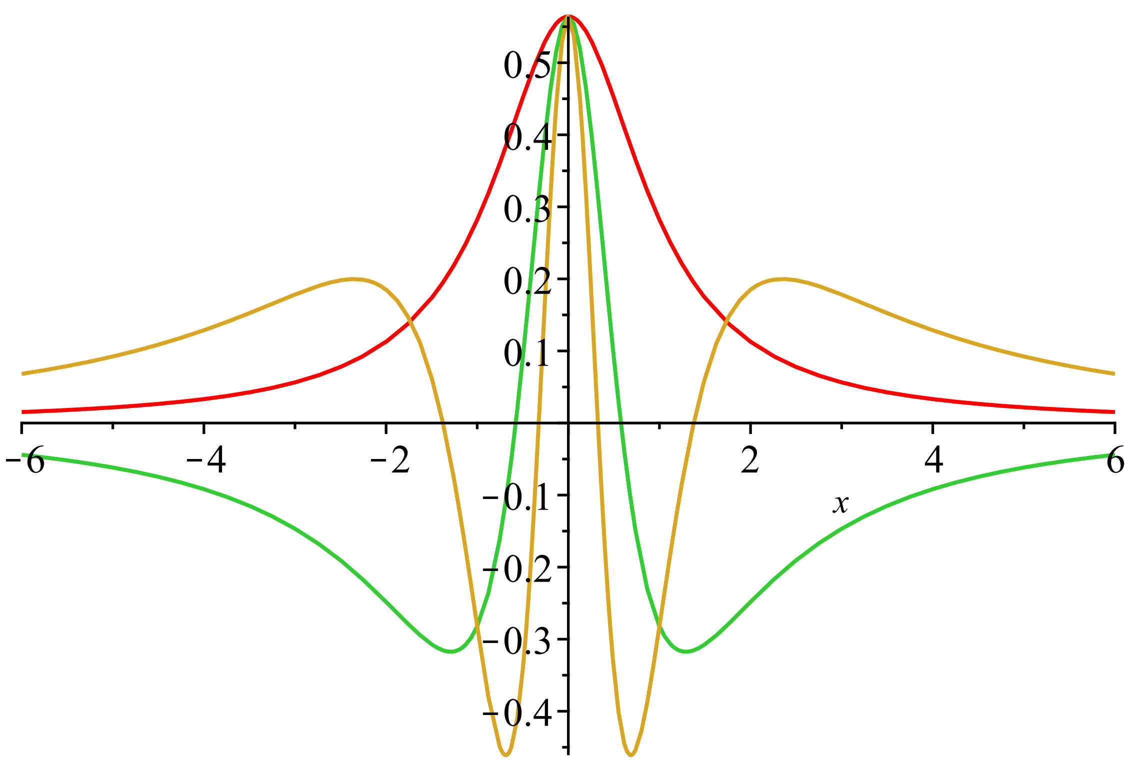
Orthogonality checking

> $\int_{-\infty}^{\infty} Rational(1, x)\, \overline{Rational(-2, x)}\, dx$

$$0 \qquad \textbf{(3.2.2)}$$

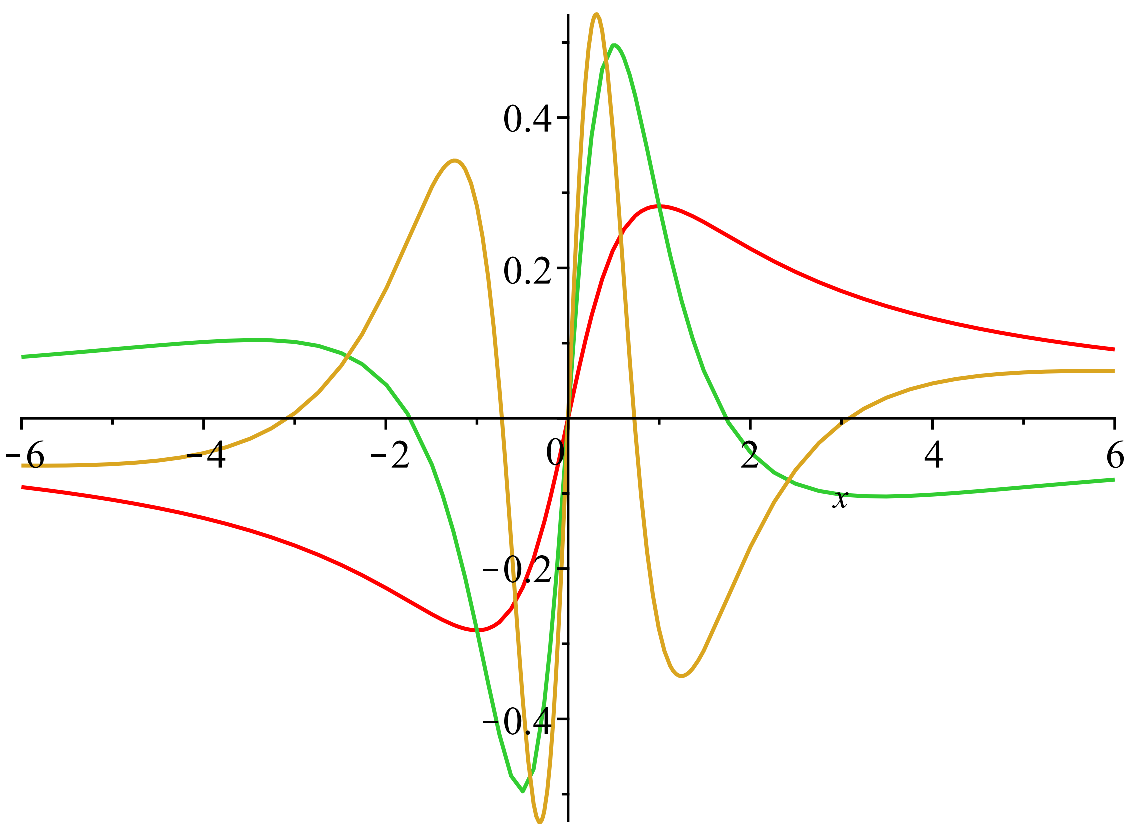
Real part of Rational functions

> $plot([\Re(Rational(0, x)), \Re(Rational(1, x)), \Re(Rational(2, x))], x = -6..6)$



Imaginary part of Rational functions

> *plot*([ℑ(*Rational*(0, *x*)), ℑ(*Rational*(1, *x*)), ℑ(*Rational*(2, *x*))], *x* = −6 ..6)



>

Walsh functions

The **Walsh(n, j, x, opt)** calling sequences computes the Walsh function with order **n** (n=0, 1, 2,...) and number **j** (j=0, 1, ..., 2^n-1). These functions are orthonormal on the interval [0,1]. They satisfy the following:

$$\int_0^1 Walsh(n, j, x, opt)\ Walsh(n, k, x, opt)\ \mathrm{d}x = \delta_{jk}.$$

The option opt can contain the following equations: order = walsh (the default), paley, or hadamard.

This option arranged order of the set of Walsh functions: order = walsh corresponds to Walsh or sequency order, order = paley corresponds to Paley or dyadic order, order = hadamard corresponds to Hadamard or natural order.

Orthogonality checking

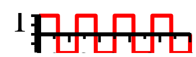
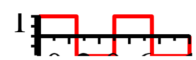
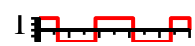
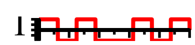
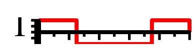
> $\int_0^1 Walsh(2, 0, x)\ Walsh(2, 1, x)\ \mathrm{d}x, \int_0^1 Walsh(2, 1, x)\ Walsh(2, 2, x)\ \mathrm{d}x$

0, 0 **(3.3.1)**

Walsh functions for different ordering

> $n := 3;$

$n := 3$ **(3.3.2)**

Function number	Walsh ordering	Paley ordering	Hadamard ordering
$j := 0$			
$j := 1$			
$j := 2$			
$j := 3$			
$j := 4$			
$j := 5$			
$j := 6$			
j			

:= 7

>

Zernike functions

This functions are widely used in optics. The following Zernike functions are called Astigmatism

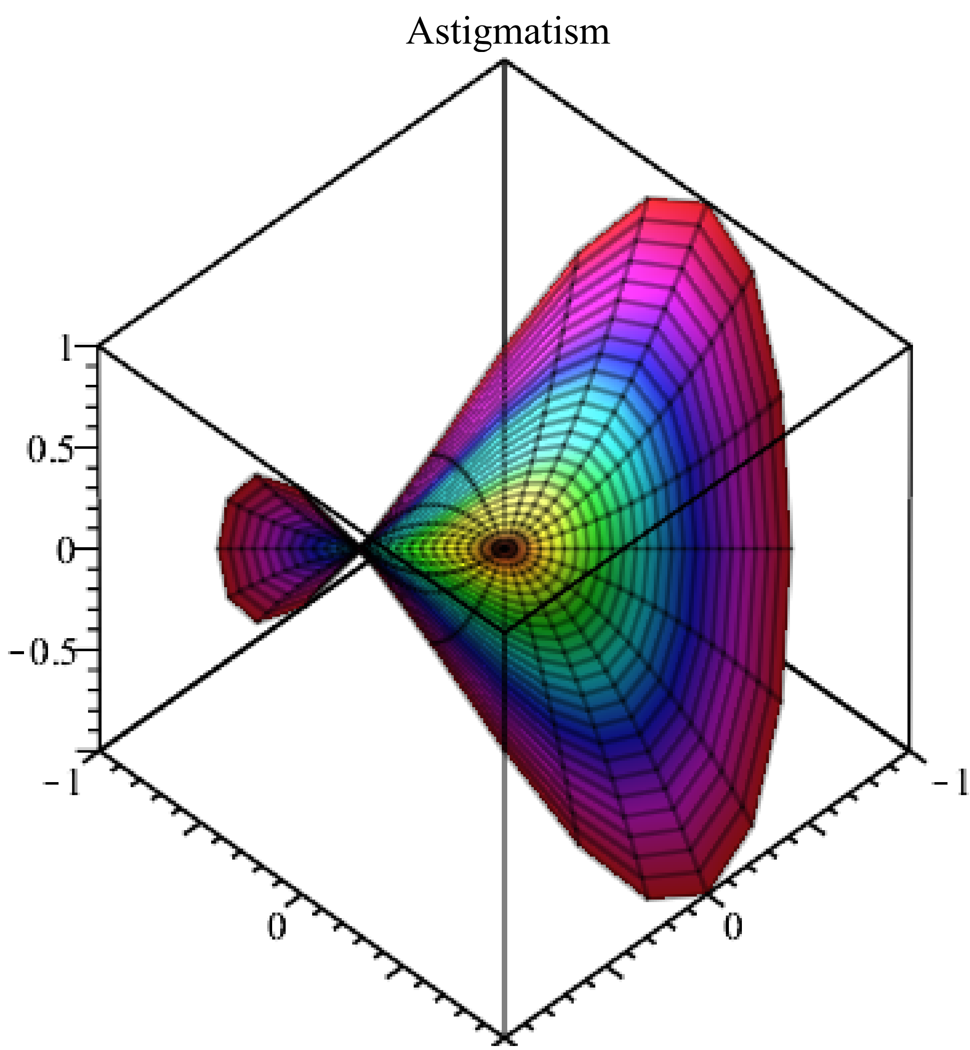
> $f := Zernike(2, 2, \rho, \phi); Zernike(2, -2, \rho, \phi);$

$$f := \rho^2 \cos(2\,\phi)$$

$$\rho^2 \sin(2\,\phi) \qquad \textbf{(3.4.1)}$$

> $addcoords(z\_cylindrical, [z, \rho, \theta], [\rho\cos(\theta), \rho\sin(\theta), z]);$

> $plot3d(f, \rho = 0..1, \phi = 0..2\,\pi, coords = z\_cylindrical, title = \text{"Astigmatism"}, color = \text{rho});$



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One-dimensional series expansions

Inert form of expansions

In order to get inert form of an expansion one can explicitly specify option output=inert or do not assign numeric value for the number of expansion terms.

> $f := x \rightarrow x + x^2$:

Inert form of expansion for the different Fourier series modifications

```
> n :='n':
  Trig := FourierSeries( f(x), x=-π..π, n,'Coefficients');
  Exp := FourierSeries( f(x), x = -π..π, n, series=exponential);
  Polar := FourierSeries( f(x), x = -π..π, n, series=polar);
  Hartley := FourierSeries( f(x), x = -π..π, n, series=hartley);
```

$$Trig := \frac{1}{3}\pi^2 + \sum_{i=1}^{n}\left(\frac{4(-1)^i\cos(i x)}{i^2} + \frac{2(-1)^{1+i}\sin(i x)}{i}\right)$$

$$Exp := \frac{1}{3}\pi^2 + \sum_{i=1}^{n}\left(\left(\frac{2(-1)^i}{i^2} - \frac{I(-1)^{1+i}}{i}\right)e^{I i x} + \left(\frac{2(-1)^i}{i^2} + \frac{I(-1)^{1+i}}{i}\right)e^{-I i x}\right)$$

$$Polar := \frac{1}{3}\pi^2 + \sum_{i=1}^{n}2\sqrt{\frac{4((-1)^i)^2}{i^4} + \frac{((-1)^{1+i})^2}{i^2}}\cos\left(i x - \arctan\left(\frac{2(-1)^{1+i}}{i}, \frac{4(-1)^i}{i^2}\right)\right)$$

$$Hartley := \frac{1}{3}\pi^2 + \sum_{i=1}^{n}\left(\frac{(-1)^{1+i}(i-2)(\cos(i x) + \sin(i x))}{i^2} + \frac{(-1)^i(i+2)(\cos(i x) - \sin(i x))}{i^2}\right) \quad \textbf{(4.1.1)}$$

The trigonometric series coefficients

> *Coefficients* := *Coefficients*;

$$Coefficients := \left[\frac{1}{3}\pi^2, \%seq\left(\left[\frac{4(-1)^i}{i^2}, \frac{2(-1)^{1+i}}{i}\right], i = 1..n\right)\right] \quad \textbf{(4.1.2)}$$

The inert forms evaluation by the **value** command

```
> n := 2;
  Trig := value(Trig);
  Exp := value(Exp);
  Polar := value(Polar);
  Hartley := value(Hartley);
  Coefficients := value(Coefficients);
```

$$n := 2$$

$$Trig := \frac{1}{3}\pi^2 - 4\cos(x) + 2\sin(x) + \cos(2x) - \sin(2x)$$

$$Exp := \frac{1}{3}\pi^2 + (-2 - I)e^{Ix} + (-2 + I)e^{-Ix} + \left(\frac{1}{2} + \frac{1}{2}I\right)e^{2Ix} + \left(\frac{1}{2} - \frac{1}{2}I\right)e^{-2Ix}$$

$$Polar := \frac{1}{3}\pi^2 - 2\sqrt{5}\cos\left(x + \arctan\left(\frac{1}{2}\right)\right) + \sqrt{2}\cos\left(2x + \frac{1}{4}\pi\right)$$

$$Hartley := \frac{1}{3}\pi^2 - 4\cos(x) + 2\sin(x) + \cos(2x) - \sin(2x)$$

$$Coefficients := \left[\frac{1}{3}\pi^2, [-4, 2], [1, -1]\right] \quad \textbf{(4.1.3)}$$

These Fourier series modifications are equivalent

```
> Trig - evalc(Exp), simplify(Trig - expand(Polar)), Trig - Hartley;
```

$$0, 0, 0 \quad \textbf{(4.1.4)}$$

but these series are not equivalent to cosine and sine series

```
> n :='n':
  Cosine := FourierSeries(f(x), x = -π..π, n, series = cosine);
  Sine := FourierSeries(f(x), x = -π..π, n, series = sine);
```

$$Cosine := \frac{1}{3}\pi^2 + \sum_{i=1}^{n} \frac{4\left(2(-1)^i\pi + (-1)^i + 2\pi - 1\right)\cos\left(\frac{1}{2} i (x+\pi)\right)}{i^2\pi}$$

$$Sine := \sum_{i=1}^{n} \left(-\frac{2\left(8(-1)^{1+i} + (-1)^i i^2\pi^2 + i^2\pi(-1)^i - i^2\pi^2 + 8 + i^2\pi\right)\sin\left(\frac{1}{2} i (x+\pi)\right)}{i^3\pi} \right) \quad \textbf{(4.1.5)}$$

Sinc series for different scale parameters

```
> f := x → e^(-|x|);
```

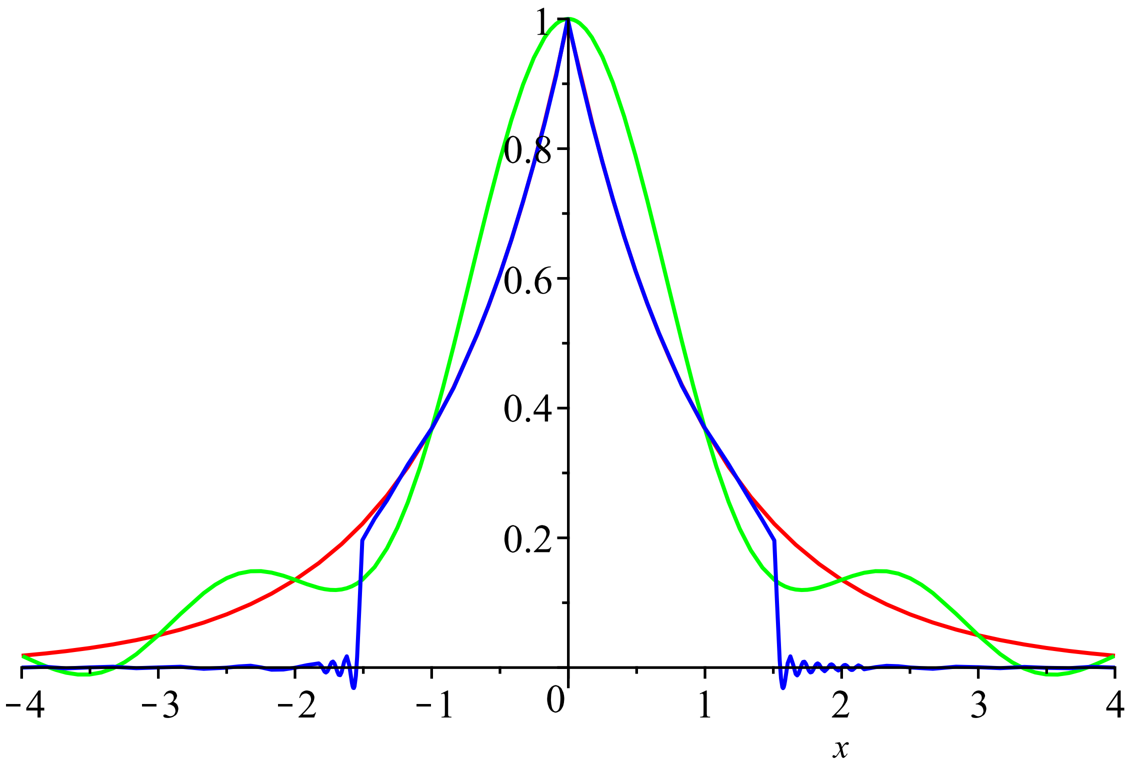
$$f := x \rightarrow e^{-|x|} \quad \textbf{(4.1.6)}$$

```
> s1 := SincSeries(f(x), x, 0, 1, 30, output = inert);       # Scale parameters is equal to 1
  s2 := SincSeries(f(x), x, 0, 1/20, 30, output = inert);    # Scale parameters is equal to 20
```

$$s1 := \frac{\sin(\pi x)}{\pi x} + \sum_{i=1}^{30} \left(\frac{e^{-|i|}\sin(\pi(x-i))}{\pi(x-i)} + \frac{e^{-|i|}\sin(\pi(x+i))}{\pi(x+i)} \right)$$

$$s2 := \frac{1}{20}\,\frac{\sin(20\pi x)}{\pi x} + \sum_{i=1}^{30} \left(\frac{e^{-\frac{1}{20}|i|}\sin(\pi(20x-i))}{\pi(20x-i)} + \frac{e^{-\frac{1}{20}|i|}\sin(\pi(20x+i))}{\pi(20x+i)} \right) \quad \textbf{(4.1.7)}$$

```
> plot([f(x), s1, s2], x = -4 .. 4, color = [red, green, blue]);
```



>

Symbolic evaluation of expansions

> $n := 8;$

$$n := 8 \qquad \textbf{(4.2.1)}$$

Bessel series

> $s := BesselSeries\left(\frac{1}{\sqrt{x}}, x = 0 \,..\, 1, n, \frac{1}{2}\right);$

$$s := \frac{4\sin(\pi x)}{\sqrt{\pi}\sqrt{\pi x}} + \frac{4}{3}\frac{\sin(3\pi x)}{\sqrt{\pi}\sqrt{\pi x}} + \frac{4}{5}\frac{\sin(5\pi x)}{\sqrt{\pi}\sqrt{\pi x}} + \frac{4}{7}\frac{\sin(7\pi x)}{\sqrt{\pi}\sqrt{\pi x}} \qquad \textbf{(4.2.2)}$$

Jacobi series

> $n := 2;$
$f := x \rightarrow \sqrt{x};$

$$n := 2$$

$$f := x \rightarrow \sqrt{x} \qquad \textbf{(4.2.3)}$$

> $s := JacobiSeries(f(x), x = 0 \,..\, 1, n, 1, 1, 'Coefficients');$
$Coefficients := Coefficients;$

$$s := \frac{24}{35}\,\mathrm{JacobiP}(0, 1, 1, 2x - 1) + \frac{4}{21}\,\mathrm{JacobiP}(1, 1, 1, 2x - 1) - \frac{16}{495}\,\mathrm{JacobiP}(2, 1, 1, 2x - 1)$$

(4.2.4)

$$Coefficients := \left[\frac{24}{35}, \frac{4}{21}, -\frac{16}{495}\right] \tag{4.2.4}$$

Gegenbauer series

```
> s := GegenbauerSeries(f(x), x=0 .. 1, n, 1, 'Coefficients');
  Coefficients := Coefficients;
```

$$s := \frac{32}{15}\,\frac{\mathrm{GegenbauerC}(0, 1, 2x-1)}{\pi} + \frac{64}{105}\,\frac{\mathrm{GegenbauerC}(1, 1, 2x-1)}{\pi} - \frac{32}{315}\,\frac{\mathrm{GegenbauerC}(2, 1, 2x-1)}{\pi}$$

$$Coefficients := \left[\frac{32}{15\,\pi}, \frac{64}{105\,\pi}, -\frac{32}{315\,\pi}\right] \tag{4.2.5}$$

Chebyshev first kind series

```
> s := ChebyshevTSeries(f(x), x=0 .. 1, n, 'Coefficients');
  Coefficients := Coefficients;
```

$$s := \frac{2\,\mathrm{ChebyshevT}(0, 2x-1)}{\pi} + \frac{4}{3}\,\frac{\mathrm{ChebyshevT}(1, 2x-1)}{\pi} - \frac{4}{15}\,\frac{\mathrm{ChebyshevT}(2, 2x-1)}{\pi}$$

$$Coefficients := \left[\frac{2}{\pi}, \frac{4}{3\,\pi}, -\frac{4}{15\,\pi}\right] \tag{4.2.6}$$

Chebyshev second kind series

```
> s := ChebyshevUSeries(f(x), x=0 .. 1, n, 'Coefficients');
  Coefficients := Coefficients;
```

$$s := \frac{32}{15}\,\frac{\mathrm{ChebyshevU}(0, 2x-1)}{\pi} + \frac{64}{105}\,\frac{\mathrm{ChebyshevU}(1, 2x-1)}{\pi} - \frac{32}{315}\,\frac{\mathrm{ChebyshevU}(2, 2x-1)}{\pi}$$

$$Coefficients := \left[\frac{32}{15\,\pi}, \frac{64}{105\,\pi}, -\frac{32}{315\,\pi}\right] \tag{4.2.7}$$

Legendre series

```
> s := LegendreSeries(f(x), x=0 .. 1, n, 0, 'Coefficients');
  Coefficients := Coefficients;
```

$$s := \frac{4}{15} + \frac{4}{5}\,x - \frac{2}{21}\,\mathrm{LegendreP}(2, 2x-1)$$

$$Coefficients := \left[\frac{2}{3}, \frac{2}{5}, -\frac{2}{21}\right] \tag{4.2.8}$$

Laguerre series

```
> s := LaguerreSeries(sqrt(x) e^(-x), x, 0, 1, n, 0, 'Coefficients');
  Coefficients := Coefficients;
```

$$s := \frac{1}{8}\,\sqrt{2}\,\sqrt{\pi}\,\mathrm{LaguerreL}(0, x) + \frac{1}{32}\,\sqrt{2}\,\sqrt{\pi}\,\mathrm{LaguerreL}(1, x)$$

$$-\frac{1}{256}\sqrt{2}\sqrt{\pi}\,\mathrm{LaguerreL}(2,x)$$

$$Coefficients := \left[\frac{1}{8}\sqrt{2}\sqrt{\pi}, \frac{1}{32}\sqrt{2}\sqrt{\pi}, -\frac{1}{256}\sqrt{2}\sqrt{\pi}\right] \tag{4.2.9}$$

Hermite series

> $s := HermiteSeries\left(x\,\mathrm{e}^{-x^2}, x, 0, 1, 6, 'Coefficients'\right);$
> $Coefficients := Coefficients;$

$$s := \frac{1}{8}\sqrt{2}\,\mathrm{HermiteH}(1,x) - \frac{1}{64}\sqrt{2}\,\mathrm{HermiteH}(3,x) + \frac{1}{1024}\sqrt{2}\,\mathrm{HermiteH}(5,x)$$

$$Coefficients := \left[0, \frac{1}{8}\sqrt{2}, 0, -\frac{1}{64}\sqrt{2}, 0, \frac{1}{1024}\sqrt{2}, 0\right] \tag{4.2.10}$$

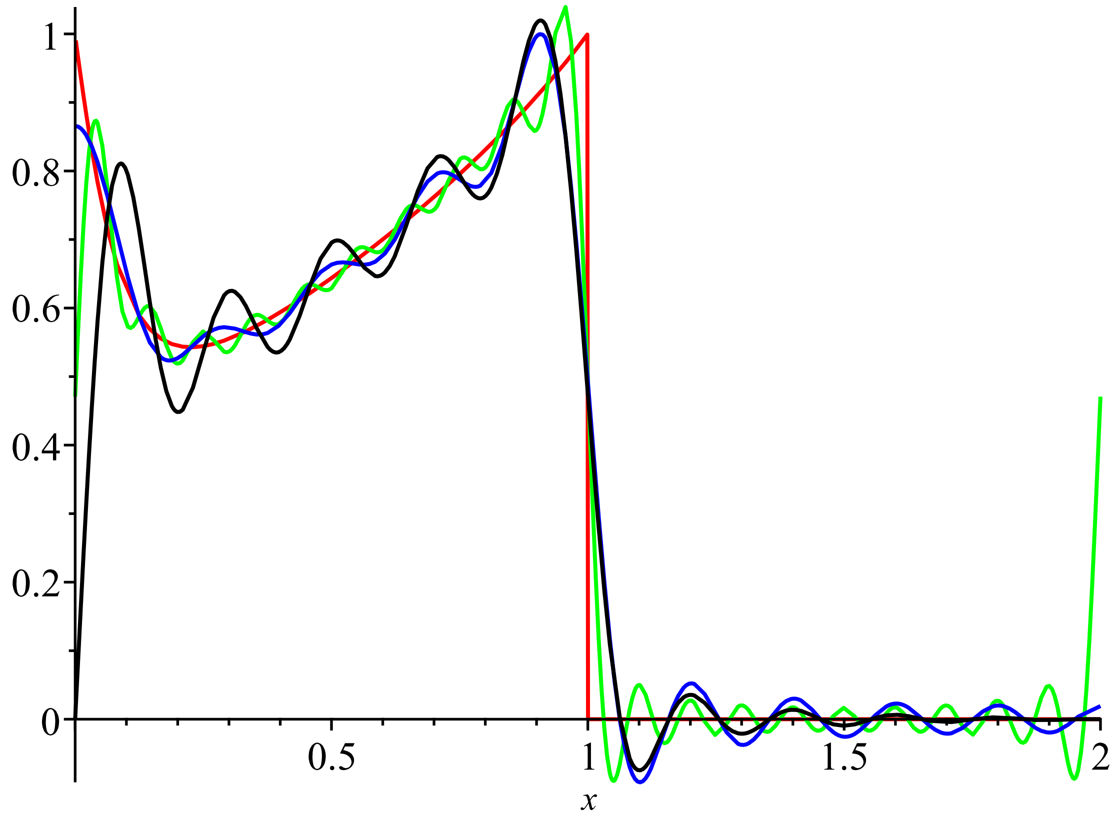
>

Numeric evaluation of expansions

Trigonometric, cosine and sine Fourier series

> $f := x \rightarrow \begin{cases} x^{x^{x^{x^{x^{x}}}}} & 0 \le x \textbf{ and } x < 1 \\ 0 & otherwise \end{cases} :$

> $Trig := FourierSeries(f(x), x = 0..2, 20, output = numeric) :$
> $Cosine := FourierSeries(f(x), x = 0..2, 20, output = numeric, series = cosine) :$
> $Sine := FourierSeries(f(x), x = 0..2, 20, output = numeric, series = sine) :$
> $plot([f(x), Trig, Cosine, Sine], x = 0..2, color = [red, green, blue, black]);$

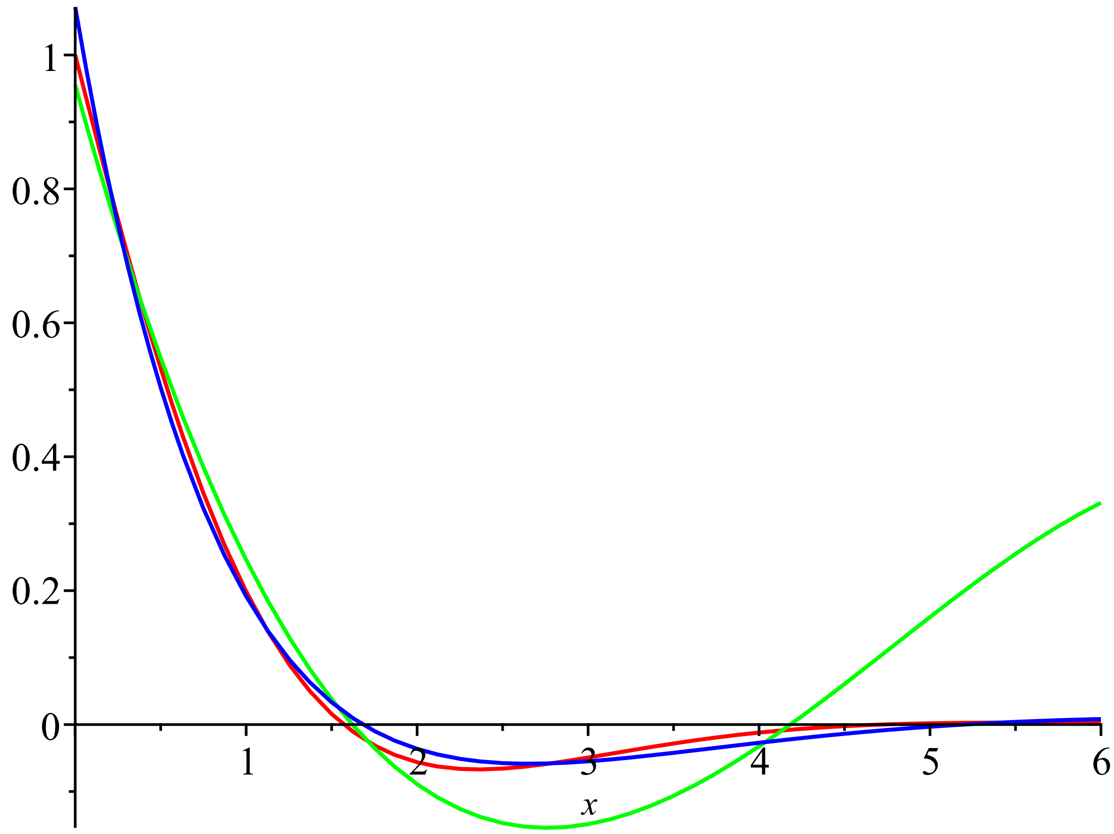


Laguerre series for different series kinds

> $f := x \to e^{-x} \cos(x)$

$$f := x \to e^{-x} \cos(x) \qquad \textbf{(4.3.1)}$$

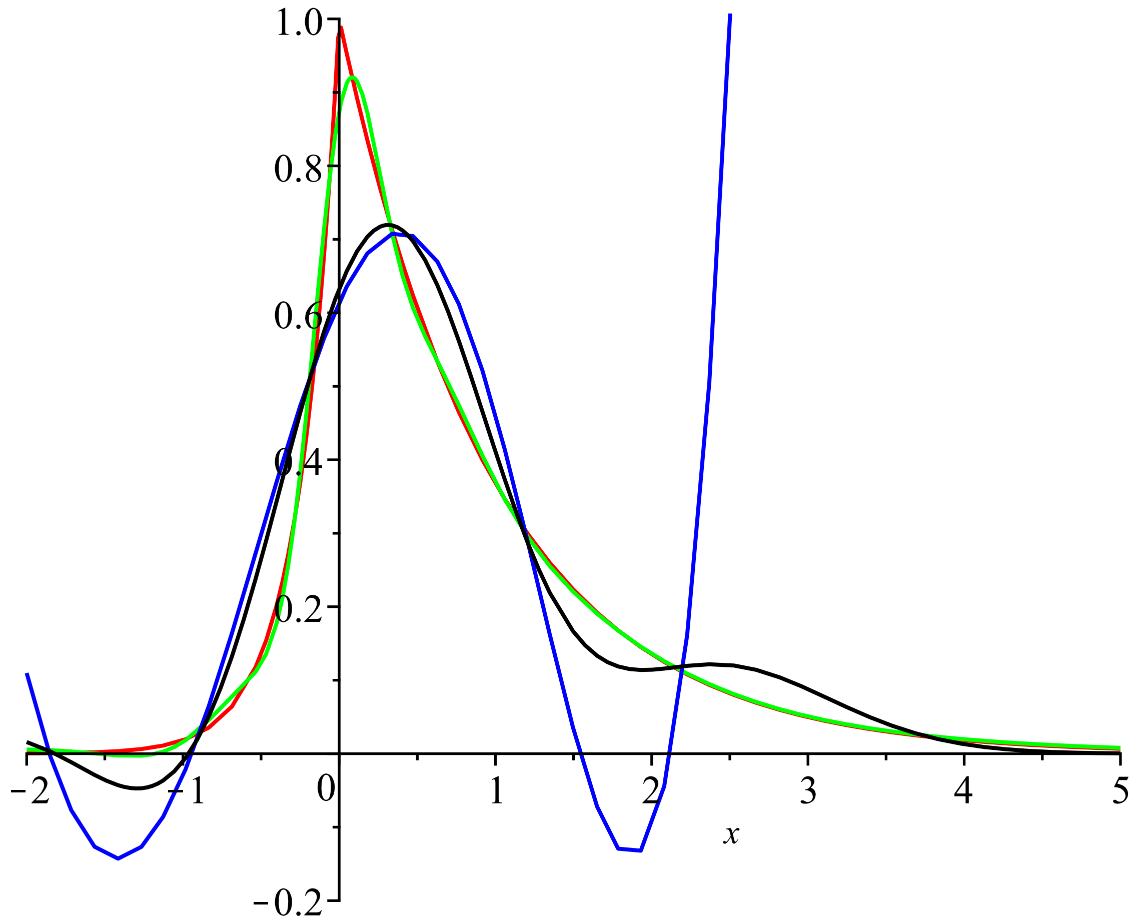
> *Polynomial* := *LaguerreSeries*($f(x)$, x, 0, 1, 3, 0, *output* = *numeric*) :
Function := *LaguerreSeries*($f(x)$, x, 0, 1, 3, 0, *output* = *numeric*, *series* = *function*) :
plot([$f(x)$, *Polynomial*, *Function*], x = 0 ..6, *color* = [*red*, *green*, *blue*]);



Rational series vs Hermite polynomial and Hermite function series kinds

> $f := x \rightarrow \begin{cases} e^{-x} & 0 \leq x \\ e^{4x} & otherwise \end{cases}$:

> $Rat := RationalSeries(f(x), x, 0, 1, 5, output = numeric)$:
$HermitPol := HermiteSeries(f(x), x, 0, 1, 5, output = numeric)$:
$HermitFun := HermiteSeries(f(x), x, 0, 1, 5, output = numeric, series = function)$:
$plot([f(x), Rat, HermitPol, HermitFun], x = -2 ..5, view = -0.2 ..1, color = [red, green, blue, black])$;



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Multi-dimensional series expansions

Inert form of expansions

Two-dimensional Fourier sine series

> $f := (x, y) \rightarrow x\, e^y;$

$$f := (x, y) \rightarrow x\, e^y \tag{5.1.1}$$

> $n := 'n':$
$s := FourierSeries\left(f(x, y), [x = 0 .. 3, y = 0 .. 3], [n_1, n_2], 'Coefficients', series = sine\right);$
$Coefficients := Coefficients;$

$$s := \sum_{k_1 = 1}^{n_1} \left(\sum_{k_2 = 1}^{n_2} \left(- \frac{12\,(-1)^{1+k_1}\, k_2 \left((-1)^{k_2} e^3 - 1\right) \prod_{i=1}^{2} \sin\left(\frac{k_i \pi\, ([x, y]_i - [0, 0]_i)}{[3, 3]_i}\right)}{k_1 \left(9 + k_2^2 \pi^2\right)} \right) \right)$$

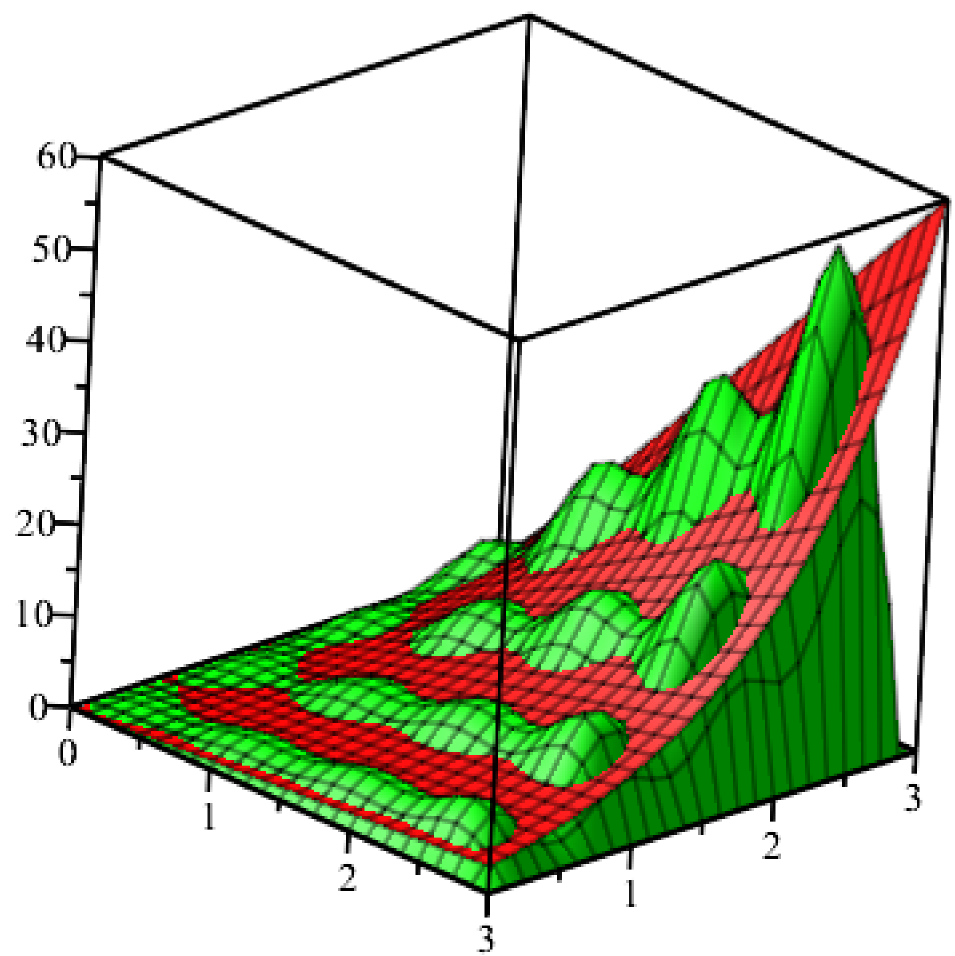
$$Coefficients := \left[\%seq\left(\left[\%seq\left(- \frac{12\,(-1)^{1+k_1}\, k_2 \left((-1)^{k_2} e^3 - 1\right)}{k_1 \left(9 + k_2^2 \pi^2\right)}, k_2 = 1 .. n_2\right)\right], k_1 = 1\right.\right. \tag{5.1.2}$$

$..n_1 \big) \big]$

> $n_1, n_2 := 7, 7;$
> $s := value(s) :$

$$n_1, n_2 := 7, 7 \quad (5.1.3)$$

> $plot3d([f(x, y), s], x = 0..3, y = 0..3, color = [red, green])$



Two-dimensional Sinc series

> $f := (x, y) \rightarrow e^{-x^2 - 3y^2} \cdot \ln\left(\frac{\pi}{2} + \sin(10 \cdot x \cdot y^2)\right);$

$$f := (x, y) \rightarrow e^{-x^2 - 3y^2} \ln\left(\frac{1}{2}\pi + \sin(10\, x\, y^2)\right) \quad (5.1.4)$$

> $n := 'n':$
> $s := SincSeries\left(f(x, y), [x, y], [0, 0], \left[\frac{1}{10}, \frac{1}{10}\right], [n_1, n_2], 'Coefficients'\right);$
> $Coefficients := Coefficients;$

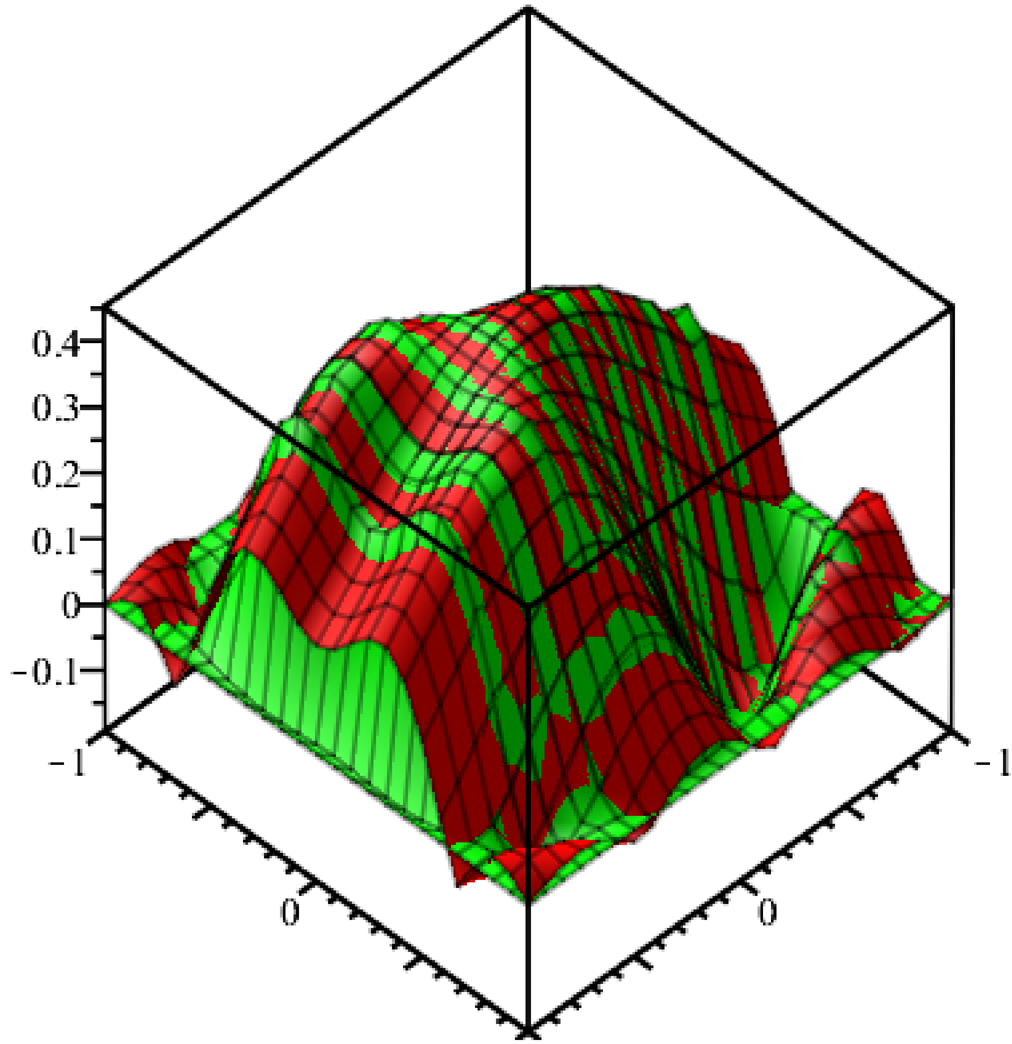
$$s := \sum_{k_1 = -n_1}^{n_1} \left(\sum_{k_2 = -n_2}^{n_2} e^{-\frac{1}{100} k_1^2 - \frac{3}{100} k_2^2} \ln\left(\frac{1}{2}\pi + \sin\left(\frac{1}{100} k_1 k_2^2\right)\right) \prod_{i=1}^{2} \left(\frac{\sin\left(\pi \left(\frac{[x, y]_i - [0, 0]_i}{\left[\frac{1}{10}, \frac{1}{10}\right]_i} - k_i\right)\right)}{\pi \left(\frac{[x, y]_i - [0, 0]_i}{\left[\frac{1}{10}, \frac{1}{10}\right]_i} - k_i\right)} \right) \right)$$

$$Coefficients := \left[\%seq\left(\left[\%seq\left(e^{-\frac{1}{100} k_1^2 - \frac{3}{100} k_2^2} \ln\left(\frac{1}{2}\pi + \sin\left(\frac{1}{100} k_1 k_2^2\right)\right), k_2 = -n_2 ..n_2 \right) \right], k_1 = -n_1 ..n_1 \right) \right] \quad \textbf{(5.1.5)}$$

> $n_1, n_2 := 8, 8;$
> $s := value(s) :$

$$n_1, n_2 := 8, 8 \quad \textbf{(5.1.6)}$$

> $plot3d([f(x, y), s], x = -1 ..1, y = -1 ..1, color = [red, green]);$



Two-dimensional mixed Fourier sine - Chebyshev second kind series

> $f := (x, y) \rightarrow \text{sqrt}(x\, y);$

$$f := (x, y) \rightarrow \sqrt{x\, y} \quad \textbf{(5.1.7)}$$

> $n :=' n'$:
$s := MixedSeries\big(f(x, y), FourierSeries, [x = 0..1, n_1, series = sine], ChebyshevUSeries, [y = 0..1, n_2], output = inert\big);$

$$s := \sum_{k_2=0}^{n_2} \left(\int_0^1 \sum_{k_1=1}^{n_1} \frac{1}{\pi^2 k_1^{(3/2)}} \left(8 \,\text{ChebyshevU}(k_2, 2y-1) \sin(k_1 \pi x) \sqrt{y} \left(-2\sqrt{-y^2+y} \,(\quad \textbf{(5.1.8)}$$

$$-1)^{k_1} \sqrt{k_1} + \text{FresnelC}\left(\sqrt{2}\sqrt{k_1}\right)\sqrt{-2y^2+2y}\right)\right) \mathrm{d}y \right) \text{ChebyshevU}(k_2, 2y-1)$$

> $n_1, n_2 := 1, 1;$
$s := value(s);$

$$n_1, n_2 := 1, 1$$

$$s := \frac{32}{15} \frac{\sin(\pi x) \left(2 + \mathrm{FresnelC}(\sqrt{2}) \sqrt{2}\right) \mathrm{ChebyshevU}(0, 2y - 1)}{\pi^2} + \frac{64}{105} \frac{\sin(\pi x) \left(2 + \mathrm{FresnelC}(\sqrt{2}) \sqrt{2}\right) \mathrm{ChebyshevU}(1, 2y - 1)}{\pi^2} \quad (5.1.9)$$

Two-dimensional mixed Sinc - Hermite series

> $f := (x, y) \rightarrow \exp(-x^2 y^2);$

$$f := (x, y) \rightarrow \mathrm{e}^{-x^2 y^2} \quad (5.1.10)$$

> $n_1, n_2 := 3, 3;$
> $s := MixedSeries(f(x, y), SincSeries, [x, 0, 1, n_1], HermiteSeries, [y, 0, 1, n_2], output = inert);$

$$n_1, n_2 := 3, 3$$

$$s := \sum_{k_2=0}^{3} \left(\int_{-\infty}^{\infty} \frac{1}{k_2!\, \pi^{3/2}\, x} \left(\mathrm{HermiteH}(k_2, y)\, \mathrm{e}^{-y^2}\, 2^{-k_2} \left(\sum_{k_1=1}^{3} \frac{1}{x^2 - k_1^2} \left(\mathrm{e}^{-k_1^2 y^2}\, x \left(\sin(\pi x - \pi k_1)\, x + \sin(\pi x - \pi k_1)\, k_1 + x \sin(\pi x + \pi k_1) - k_1 \sin(\pi x + \pi k_1) \right) \right) + \sin(\pi x) \right) \right) \mathrm{d}y \right) \mathrm{HermiteH}(k_2, y) \quad (5.1.11)$$

> $s := value(s);$

$$s := -\frac{1}{5} \frac{1}{\pi x (x^6 - 14 x^4 + 49 x^2 - 36)} \Big(\sin(\pi x) \big(180 + 4 x^2 \sqrt{2} \sqrt{5} + 5 x^6 \sqrt{2} - 5 x^6 - 5 x^4 \sqrt{2} \sqrt{5} + x^6 \sqrt{2} \sqrt{5} - 2 x^6 \sqrt{5} - 65 x^4 \sqrt{2} + 180 x^2 \sqrt{2} + 70 x^4 - 245 x^2 + 20 x^4 \sqrt{5} - 18 x^2 \sqrt{5}\big) \mathrm{HermiteH}(0, y)\Big) + \frac{1}{200} \frac{1}{\pi (x^6 - 14 x^4 + 49 x^2 - 36)} \Big(\sin(\pi x)\, x \big(900 \sqrt{2} + 9 x^4 \sqrt{2} \sqrt{5} - 45 x^2 \sqrt{2} \sqrt{5} + 36 \sqrt{2} \sqrt{5} - 16 x^4 \sqrt{5} - 144 \sqrt{5} + 160 x^2 \sqrt{5} + 25 x^4 \sqrt{2} - 325 x^2 \sqrt{2}\big) \mathrm{HermiteH}(2, y)\Big) \quad (5.1.12)$$

>

Symbolic evaluation of expansions

> $n_1, n_2 := 2, 2;$

$$n_1, n_2 := 2, 2 \quad (5.2.1)$$

Two-dimensional Fourier cosine series

> $f := (x, y) \to x\, e^{y};$

$$f := (x, y) \to x\, e^{y} \quad (5.2.2)$$

> $s := FourierSeries\left(f(x, y), [x = 0 .. 1, y = 0 .. 1], [n_1, n_2], 'Coefficients', series = cosine\right);$
$Coefficients := Coefficients;$

$$s := \frac{1}{2} e - \frac{1}{2} - \frac{(e + 1)\cos(\pi y)}{1 + \pi^2} + \frac{(e - 1)\cos(2\pi y)}{1 + 4\pi^2} - \frac{4(e - 1)\cos(\pi x)}{\pi^2} + \frac{8(e + 1)\cos(\pi x)\cos(\pi y)}{\pi^2(1 + \pi^2)} - \frac{8(e - 1)\cos(\pi x)\cos(2\pi y)}{\pi^2(1 + 4\pi^2)}$$

$$Coefficients := \left[\left[\frac{1}{2} e - \frac{1}{2}, -\frac{e + 1}{1 + \pi^2}, \frac{e - 1}{1 + 4\pi^2}\right], \left[-\frac{4(e - 1)}{\pi^2}, \frac{8(e + 1)}{\pi^2(1 + \pi^2)}, -\frac{8(e - 1)}{\pi^2(1 + 4\pi^2)}\right], [0, 0, 0]\right] \quad (5.2.3)$$

Two-dimensional Laguerre series

> $f := (x, y) \to y\, e^{-x} + x\, e^{-y};$

$$f := (x, y) \to y\, e^{-x} + x\, e^{-y} \quad (5.2.4)$$

> $s := LaguerreSeries\left(f(x, y), [x, y], [0, 0], [1, 1], [n_1, n_2], 0, 'Coefficients'\right);$
$Coefficients := Coefficients;$

$$s := \mathrm{LaguerreL}(0, x)\,\mathrm{LaguerreL}(0, y) - \frac{1}{4}\mathrm{LaguerreL}(0, x)\,\mathrm{LaguerreL}(1, y) + \frac{1}{8}\mathrm{LaguerreL}(0, x)\,\mathrm{LaguerreL}(2, y) - \frac{1}{4}\mathrm{LaguerreL}(1, x)\,\mathrm{LaguerreL}(0, y) - \frac{1}{2}\mathrm{LaguerreL}(1, x)\,\mathrm{LaguerreL}(1, y) - \frac{1}{8}\mathrm{LaguerreL}(1, x)\,\mathrm{LaguerreL}(2, y) + \frac{1}{8}\mathrm{LaguerreL}(2, x)\,\mathrm{LaguerreL}(0, y) - \frac{1}{8}\mathrm{LaguerreL}(2, x)\,\mathrm{LaguerreL}(1, y)$$

$$Coefficients := \left[\left[1, -\frac{1}{4}, \frac{1}{8}\right], \left[-\frac{1}{4}, -\frac{1}{2}, -\frac{1}{8}\right], \left[\frac{1}{8}, -\frac{1}{8}, 0\right]\right] \quad (5.2.5)$$

Two-dimensional Hermite polynomial, Hermite function and mixed Hermite polynomial - Hermite function series kinds

> $f := (x, y) \to y\, e^{-x^2} + x\, e^{-y^2};$

$$f := (x, y) \to y\, e^{-x^2} + x\, e^{-y^2} \quad (5.2.6)$$

> $Polynomial := HermiteSeries\left(f(x, y), [x, y], [0, 0], [1, 1], [n_1, n_2], 'Coefficients'\right);$
$Coefficients := Coefficients;$

$$Polynomial := \frac{1}{4}\sqrt{2}\,\mathrm{HermiteH}(0, x)\,\mathrm{HermiteH}(1, y) + \frac{1}{4}\sqrt{2}\,\mathrm{HermiteH}(1,$$

$$x)\ \mathrm{HermiteH}(0,y) - \frac{1}{32}\sqrt{2}\ \mathrm{HermiteH}(1,x)\ \mathrm{HermiteH}(2,y)$$

$$- \frac{1}{32}\sqrt{2}\ \mathrm{HermiteH}(2,x)\ \mathrm{HermiteH}(1,y)$$

$$Coefficients := \left[\left[0, \frac{1}{4}\sqrt{2}, 0\right], \left[\frac{1}{4}\sqrt{2}, 0, -\frac{1}{32}\sqrt{2}\right], \left[0, -\frac{1}{32}\sqrt{2}, 0\right]\right] \quad \textbf{(5.2.7)}$$

> $Function := HermiteSeries\left(f(x,y), [x,y], [0,0], [1,1], [n_1, n_2], 'Coefficients', series = function\right);$
> $Coefficients := Coefficients;$

$$Function := \frac{2}{3}\sqrt{3}\ \mathrm{e}^{-\frac{1}{2}x^2}\ \mathrm{HermiteH}(0,x)\ \mathrm{e}^{-\frac{1}{2}y^2}\ \mathrm{HermiteH}(1,y)$$

$$+ \frac{2}{3}\sqrt{3}\ \mathrm{e}^{-\frac{1}{2}x^2}\ \mathrm{HermiteH}(1,x)\ \mathrm{e}^{-\frac{1}{2}y^2}\ \mathrm{HermiteH}(0,y)$$

$$- \frac{1}{18}\sqrt{3}\ \mathrm{e}^{-\frac{1}{2}x^2}\ \mathrm{HermiteH}(1,x)\ \mathrm{e}^{-\frac{1}{2}y^2}\ \mathrm{HermiteH}(2,y)$$

$$- \frac{1}{18}\sqrt{3}\ \mathrm{e}^{-\frac{1}{2}x^2}\ \mathrm{HermiteH}(2,x)\ \mathrm{e}^{-\frac{1}{2}y^2}\ \mathrm{HermiteH}(1,y)$$

$$Coefficients := \left[\left[0, \frac{2}{3}\sqrt{3}, 0\right], \left[\frac{2}{3}\sqrt{3}, 0, -\frac{1}{18}\sqrt{3}\right], \left[0, -\frac{1}{18}\sqrt{3}, 0\right]\right] \quad \textbf{(5.2.8)}$$

> $Mixed := MixedSeries\left(f(x,y), HermiteSeries, [x, 0, 1, n_1], HermiteSeries, [y, 0, 1, n_2, series = function]\right);$

$$Mixed := \left(\mathrm{HermiteH}(0,x) - \frac{1}{8}\mathrm{HermiteH}(2,x)\right)\mathrm{e}^{-\frac{1}{2}y^2}\ \mathrm{HermiteH}(1,y) \quad \textbf{(5.2.9)}$$

$$- \frac{1}{72}\sqrt{6}\ \mathrm{HermiteH}(1,x)\ \mathrm{e}^{-\frac{1}{2}y^2}\ \mathrm{HermiteH}(2,y) + \frac{1}{6}\sqrt{6}\ \mathrm{HermiteH}(1,$$

$$x)\ \mathrm{e}^{-\frac{1}{2}y^2}\ \mathrm{HermiteH}(0,y)$$

Zernike series

> $f := (\rho, \phi) \rightarrow \mathrm{e}^{\rho}\sin(\phi)^2;$

$$f := (\rho, \phi) \rightarrow \mathrm{e}^{\rho}\sin(\phi)^2 \quad \textbf{(5.2.10)}$$

> $s := ZernikeSeries(f(\rho,\phi), [\rho = 0..1, \phi = 0..2\pi], 4, 'Coefficients');$
> $Coefficients := Coefficients;$

$$s := 1 + (-12\,\mathrm{e} + 33)\left(-1 + 2\rho^2\right) + (6\,\mathrm{e} - 18)\,\rho^2\cos(2\phi) + (-1260\,\mathrm{e} + 3425)\left(1\right.$$

$$\left. - 6\rho^2 + 6\rho^4\right) - (850\,\mathrm{e} - 2310)\,\rho^2\left(3 - 4\rho^2\right)\cos(2\phi)$$

$$Coefficients := [[1], [0, 0, 0], [0, 0, -12\,\mathrm{e} + 33, 0, 6\,\mathrm{e} - 18], [0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, -1260\,\mathrm{e} + 3425, 0, 850\,\mathrm{e} - 2310, 0, 0]] \quad \textbf{(5.2.11)}$$

Four-dimensional mixed Zernike - two-dimensional Fourier cosine series

> $n_1, n_2, n_3 := 1, 1, 1;$

$$n_1, n_2, n_3 := 1, 1, 1 \quad \textbf{(5.2.12)}$$

> $f := (x, y, z, h) \rightarrow \frac{x^2 y \sqrt{z}}{1 + h};$

$$f := (x, y, z, h) \rightarrow \frac{x^2 y \sqrt{z}}{h + 1} \quad \textbf{(5.2.13)}$$

> $s := MixedSeries\left(f(x, y, z, h), ZernikeSeries, \left[[x = 0..1, y = 0..1], n_1\right], FourierSeries, \left[[z = 0..1, h = 0..1], \left[n_2, n_3\right], series = cosine\right]\right) :$
> $s := factor(expand(s));$

$$s := \frac{1}{60} \frac{1}{\pi^2} \left(\left(-\ln(2) + 2\cos(\pi h)\,\mathrm{Ci}(2\pi) - 2\cos(\pi h)\,\mathrm{Ci}(\pi) \right) \left(5\pi - 32\,x \sin(\pi y)\cos(\pi y)\right) \left(-2\pi + 3\sqrt{2}\,\mathrm{FresnelS}\left(\sqrt{2}\right)\cos(\pi z)\right) \right) \quad \textbf{(5.2.14)}$$

Numeric evaluation of expansions

Two-dimensional Chebyshev second kind series

> $f := (x, y) \rightarrow \sqrt{x + y}$

$$f := (x, y) \rightarrow \sqrt{x + y} \quad \textbf{(5.3.1)}$$

> $ChebyshevUSeries(f(x, y), [x = 0..1, y = 0..1], [1, 1], output = numeric)$

$$0.982272 + 0.131352\,\mathrm{ChebyshevU}(1, 2.\,y - 1.) + 0.131352\,\mathrm{ChebyshevU}(1, 2.\,x - 1.) - 0.0193253\,\mathrm{ChebyshevU}(1, 2.\,x - 1.)\,\mathrm{ChebyshevU}(1, 2.\,y - 1.) \quad \textbf{(5.3.2)}$$

Two-dimensional Rect series

> $n_1, n_2 := 6, 6;$

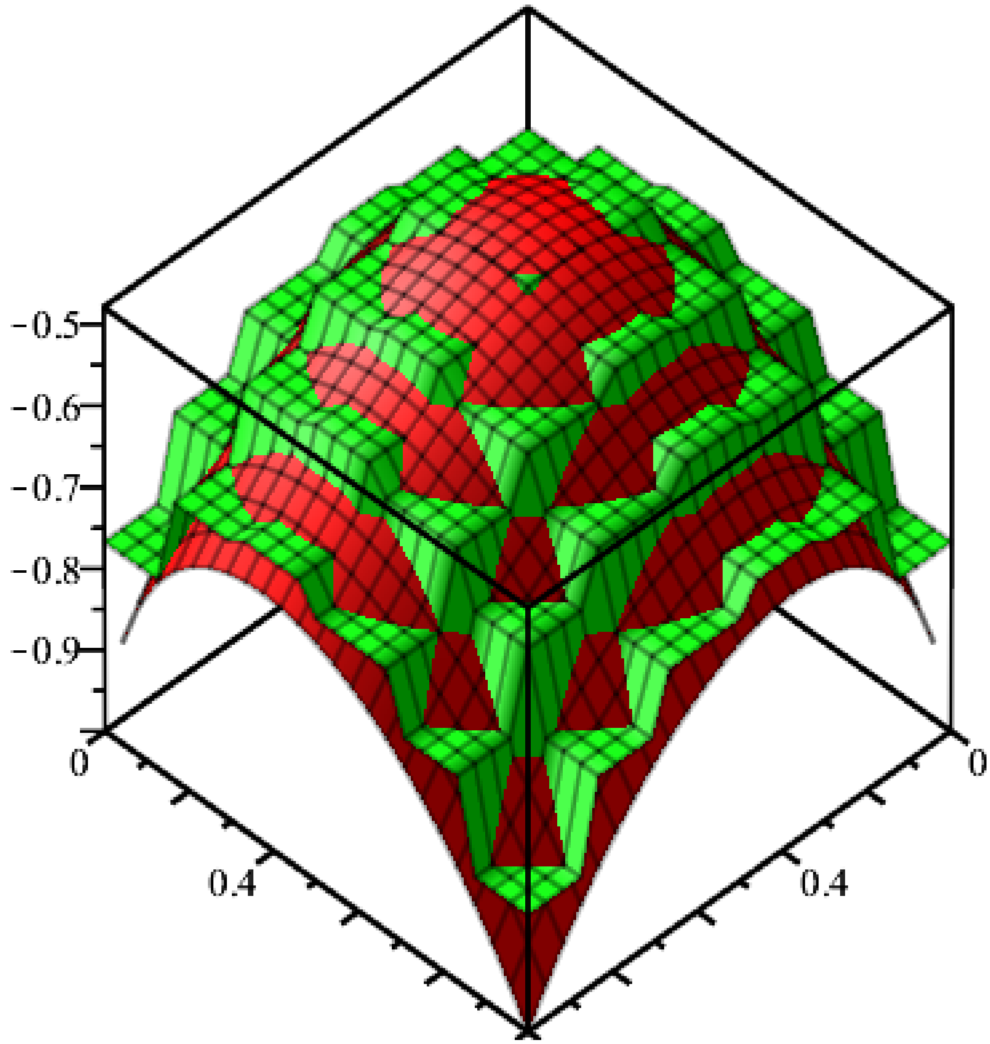
$$n_1, n_2 := 6, 6 \quad \textbf{(5.3.3)}$$

> $f := (x, y) \rightarrow -x^x y^y;$

$$f := (x, y) \rightarrow -x^x y^y \quad \textbf{(5.3.4)}$$

> $s := RectSeries\left(f(x, y), [x = 0..1, y = 0..1], \left[n_1, n_2\right], output = numeric\right) :$

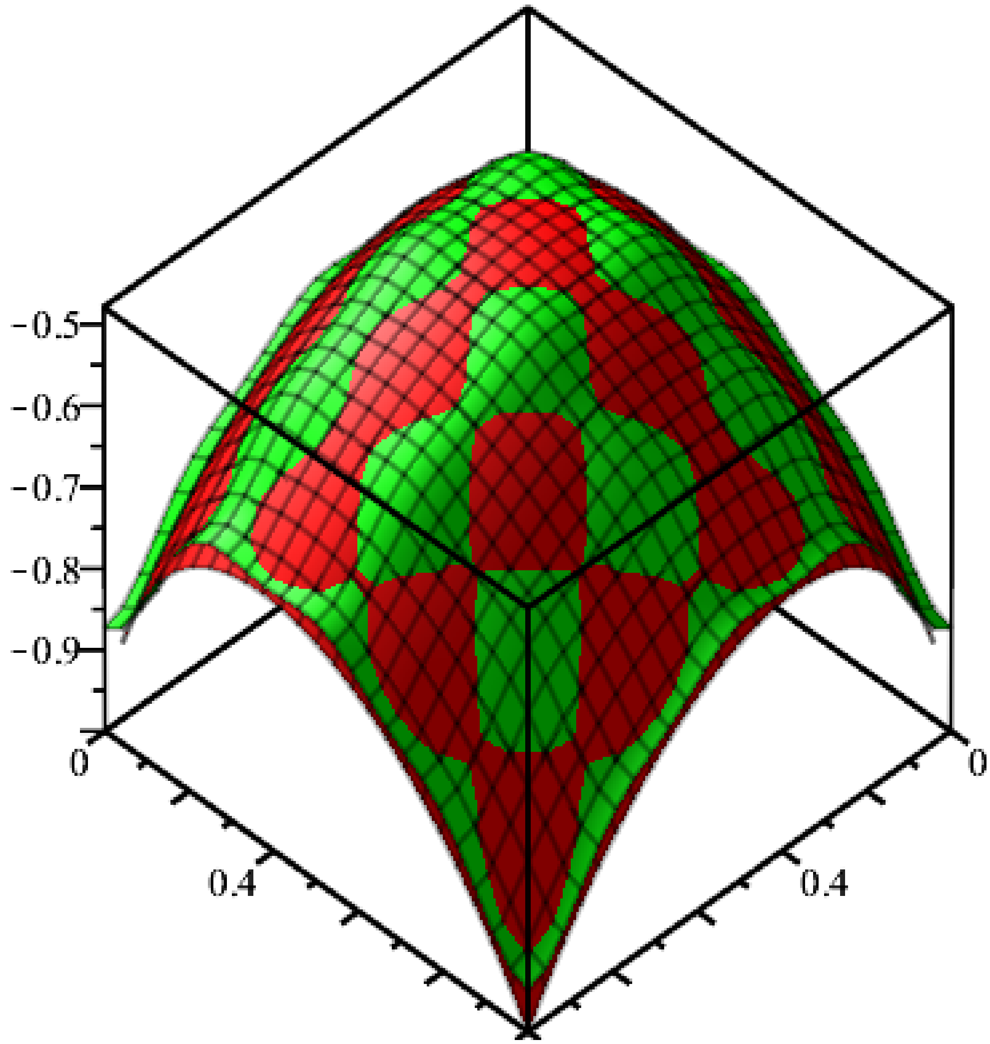
> $plot3d([f(x, y), s], x = 0..1, y = 0..1, color = [red, green]);$



Two-dimensional Fourier cosine series

> $s := FourierSeries\left(f(x, y), [x = 0..1, y = 0..1], [n_1, n_2], output = numeric, series = cosine\right):$

> $plot3d([f(x, y), s], x = 0..1, y = 0..1, color = [red, green]);$



Two-dimensional Jacobi series

> $f := (x, y) \rightarrow e^{-xy} \sin(15\, x \ln(1 + x\, y))$

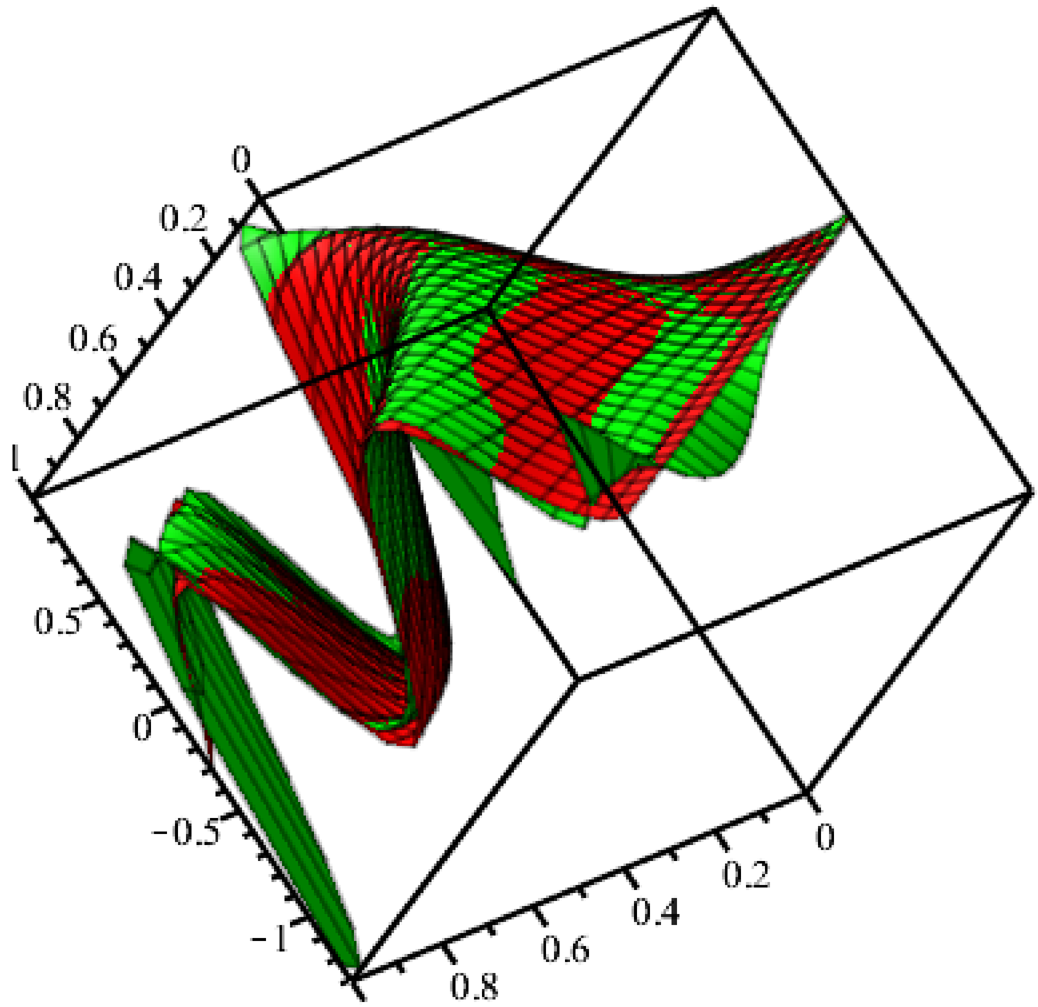
$$f := (x, y) \rightarrow e^{-xy} \sin(15\, x \ln(1 + x\, y)) \quad \textbf{(5.3.5)}$$

> $n_1, n_2 := 6, 6$

$$n_1, n_2 := 6, 6 \quad \textbf{(5.3.6)}$$

> $s := JacobiSeries\left(f(x, y),\ [x = 0..1, y = 0..1],\ [n_1, n_2],\ 1,\ 1,\ output = numeric\right)$:

> $plot3d([f(x, y), s], x = 0..1, y = 0..1, color = [red, green])$



Two-dimensional Fourier cosine series

> $f := (x, y) \rightarrow \sin\left(x\sqrt{y}\right);$

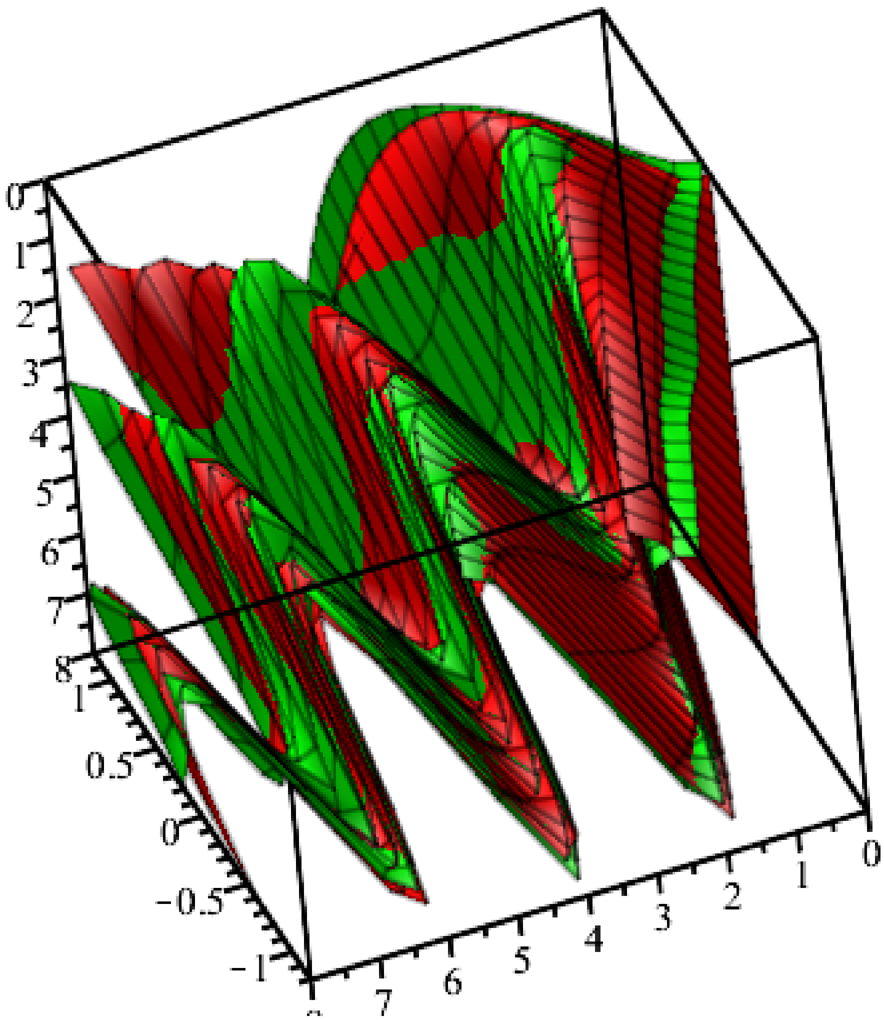
$$f := (x, y) \rightarrow \sin\left(x\sqrt{y}\right) \quad \textbf{(5.3.7)}$$

> $n_1, n_2 := 8, 8$

$$n_1, n_2 := 8, 8 \quad \textbf{(5.3.8)}$$

> $s := FourierSeries\left(f(x, y), [x = 0..8, y = 0..8], [n_1, n_2], series = cosine, output = numeric\right):$

> $plot3d([f(x, y), s], x = 0..8, y = 0..8, color = [red, green])$



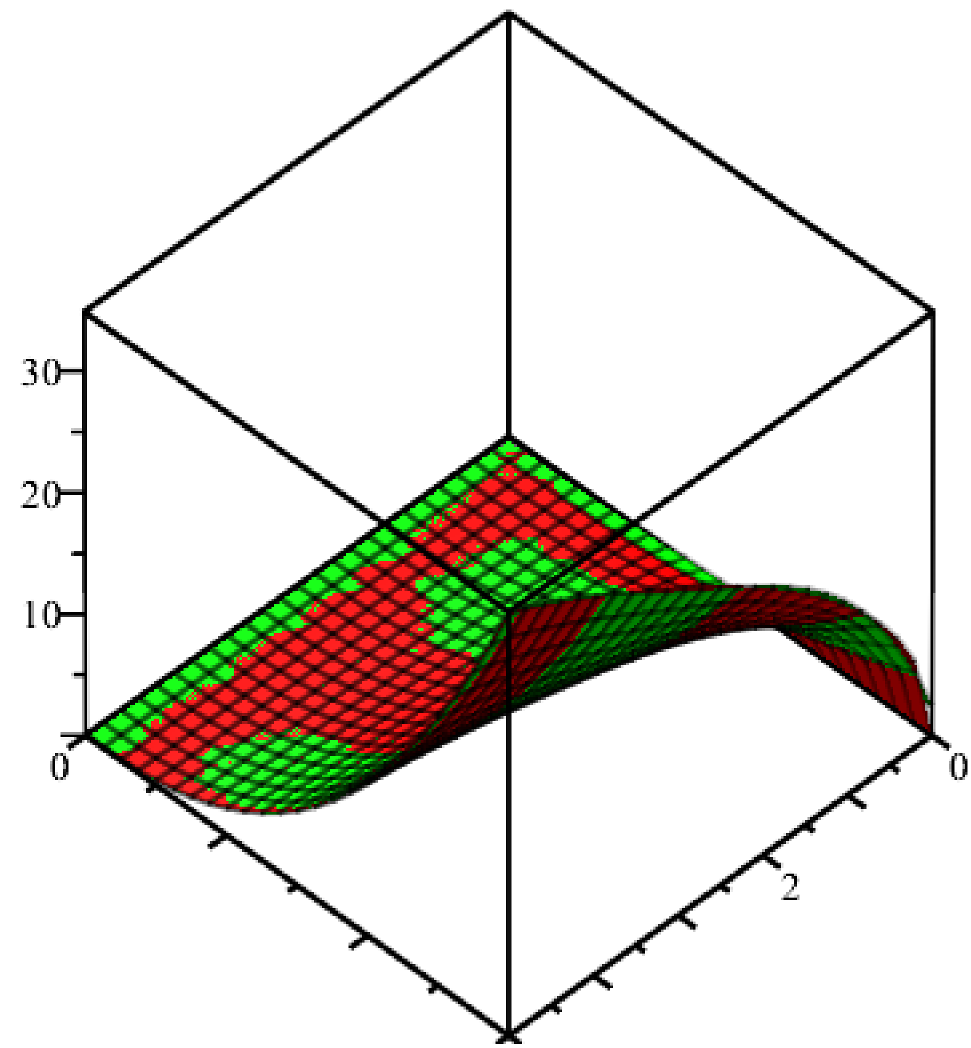
Two-dimensional mixed Chebyshev first kind - Chebyshev second kind series

> $f := (x, y) \rightarrow \sqrt{x\,y^5}$

$$f := (x, y) \rightarrow \sqrt{x\,y^5} \quad \textbf{(5.3.9)}$$

> $s := MixedSeries(f(x, y), ChebyshevTSeries, [x = 0..5, 4], ChebyshevUSeries, [y = 0..3, 4], output = numeric)$:

> $plot3d([f(x, y), s], x = 0..5, y = 0..3, color = [red, green])$;



Spherical Harmonic series

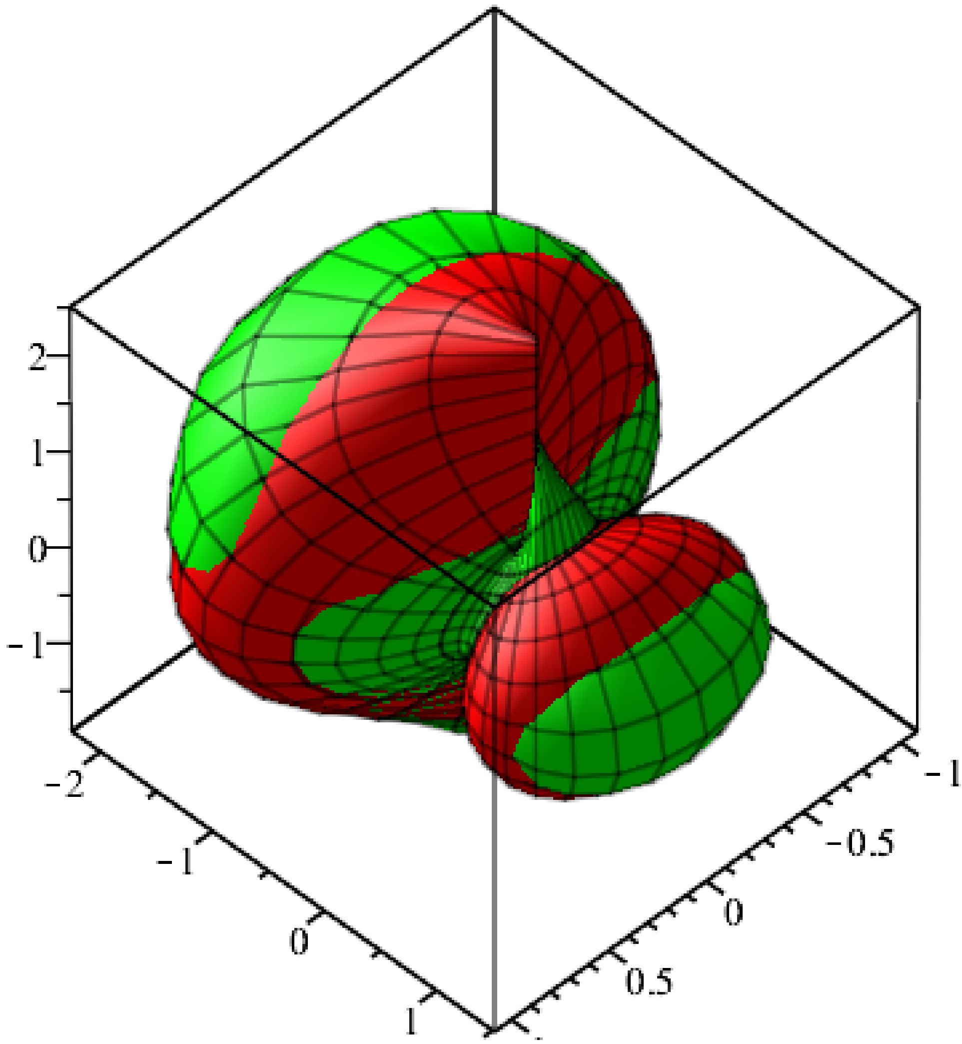
> $f := (\theta, \phi) \rightarrow \sin(\theta)\sqrt{\phi};$

$$f := (\theta, \phi) \rightarrow \sin(\theta)\sqrt{\phi} \quad \textbf{(5.3.10)}$$

> $s := SphericalSeries\left(f(\theta, \phi), [\theta = 0..\pi, \phi = 0..2\pi], 3, output = numeric, evalfIntoptions = [digits = 3]\right):$

> $s := \mathrm{Re}(s):$

> $plot3d\left([f(\theta, \phi), s], \theta = 0..\pi, \phi = 0..2\pi, color = [red, green], coords = spherical\right)$



>

Orthonormalization of functions

The GramSchmidtL2 command computes a list of orthonormal functions by using the Gram-Schmidt orthonormalization process in an L2 space of square integrable functions.

Orthonormalization of real functions

The initial list of linear-independent functions:

> $F := \left[1, x, x^{\frac{1}{2}}, x^{\frac{1}{3}}\right]$

$$F := \left[1, x, \sqrt{x}, x^{1/3}\right] \quad \textbf{(6.1)}$$

Orthonormalization of these functions on interval [0, 1] by Gram-Schmidt orthonormalization process

> $f := GramSchmidtL2(F, x = 0..1)$

$$f := \left[1, 2\left(x - \frac{1}{2}\right)\sqrt{3}, 15\left(\sqrt{x} - \frac{4}{15} - \frac{4}{5}x\right)\sqrt{2}, \frac{154}{3}\left(x^{1/3} - \frac{9}{77} + \frac{45}{154}x - \frac{90}{77}\sqrt{x}\right)\sqrt{15}\right] \quad \textbf{(6.2)}$$

Orthonormality checking

$$> \int_0^1 f_1^2\,dx, \int_0^1 f_2^2\,dx, \int_0^1 f_3^2\,dx, \int_0^1 f_4^2\,dx;$$
$$\int_0^1 f_1 f_2\,dx, \int_0^1 f_1 f_3\,dx, \int_0^1 f_1 f_4\,dx, \int_0^1 f_2 f_3\,dx, \int_0^1 f_2 f_4\,dx, \int_0^1 f_3 f_4\,dx$$

$$1, 1, 1, 1$$
$$0, 0, 0, 0, 0, 0 \tag{6.3}$$

Weighted orthonormalization

$$> F := [1, x, x^2, x^3];$$
$$w := e^{-|x|}; \quad \# \textit{ Weight function}$$

$$F := [1, x, x^2, x^3]$$
$$w := e^{-|x|} \tag{6.4}$$

$$> f := GramSchmidtL2(F, x = -\infty..\infty, weight = w)$$

$$f := \left[\frac{1}{2}\sqrt{2}, \frac{1}{2}x, \frac{1}{20}(x^2-2)\sqrt{10}, \frac{1}{72}(x^3-12x)\sqrt{6}\right] \tag{6.5}$$

Orthonormality checking

$$> \int_{-\infty}^{\infty} w f_1^2\,dx, \int_{-\infty}^{\infty} w f_2^2\,dx, \int_{-\infty}^{\infty} w f_3^2\,dx, \int_{-\infty}^{\infty} w f_4^2\,dx;$$
$$\int_{-\infty}^{\infty} w f_1 f_2\,dx, \int_{-\infty}^{\infty} w f_1 f_3\,dx, \int_{-\infty}^{\infty} w f_1 f_4\,dx, \int_{-\infty}^{\infty} w f_2 f_3\,dx, \int_{-\infty}^{\infty} w f_2 f_4\,dx, \int_{-\infty}^{\infty} w f_3 f_4\,dx$$

$$1, 1, 1, 1$$
$$0, 0, 0, 0, 0, 0 \tag{6.6}$$

Orthonormalization of complex functions

$$> F := [I, 1 + Ix, (1 + Ix)^2]$$

$$F := [I, 1 + Ix, (1 + Ix)^2] \tag{6.7}$$

$$> f := GramSchmidtL2(F, x = 0..1)$$

$$f := \left[I, 2\left(-\frac{1}{2}I + Ix\right)\sqrt{3}, 6\left((1+Ix)^2 - \frac{2}{3} - I - 2\left(-\frac{1}{2}I + Ix\right)\sqrt{3}\left(\frac{1}{3}\sqrt{3} + \frac{1}{6}I\sqrt{3}\right)\right)\sqrt{5}\right] \tag{6.8}$$

Orthonormality checking

$$> \int_0^1 f_1\overline{f_1}\,dx, \int_0^1 f_2\overline{f_2}\,dx, \int_0^1 f_3\overline{f_3}\,dx;$$
$$\int_0^1 f_1\overline{f_2}\,dx, \int_0^1 f_1\overline{f_3}\,dx, \int_0^1 f_2\overline{f_3}\,dx$$

$$1, 1, 1$$
$$0, 0, 0 \tag{6.9}$$

Orthonormalization of multivariate functions

> $F := [1, x+y, (x+y)^2, (x+y)^3]$

$$F := [1, x+y, (x+y)^2, (x+y)^3] \quad (6.10)$$

> $f := GramSchmidtL2(F, [x=0..1, y=0..1]);$

$$f := \left[1, (x+y-1)\sqrt{6}, \frac{6}{7}\left((x+y)^2 + \frac{5}{6} - 2x - 2y\right)\sqrt{35}, \frac{10}{19}\left((x+y)^3 - \frac{3}{5} + \frac{13}{5}x + \frac{13}{5}y - 3(x+y)^2\right)\sqrt{399}\right] \quad (6.11)$$

Orthonormality checking

> $\int_0^1\int_0^1 f_1^2\,dx\,dy, \int_0^1\int_0^1 f_2^2\,dx\,dy, \int_0^1\int_0^1 f_3^2\,dx\,dy, \int_0^1\int_0^1 f_4^2\,dx\,dy;$
$\int_0^1\int_0^1 f_1 f_2\,dx\,dy, \int_0^1\int_0^1 f_1 f_3\,dx\,dy, \int_0^1\int_0^1 f_1 f_4\,dx\,dy, \int_0^1\int_0^1 f_2 f_3\,dx\,dy, \int_0^1\int_0^1 f_2 f_4\,dx\,dy, \int_0^1\int_0^1 f_3 f_4\,dx\,dy$

$$1, 1, 1, 1$$
$$0, 0, 0, 0, 0, 0 \quad (6.12)$$

Numeric orthonormalization

> $F := \left[x, x^x, x^{x^x}, x^{x^{x^x}}\right]$

$$F := \left[x, x^x, x^{x^x}, x^{x^{x^x}}\right] \quad (6.13)$$

> $f := GramSchmidtL2(F, x=0..1, output=numeric)$

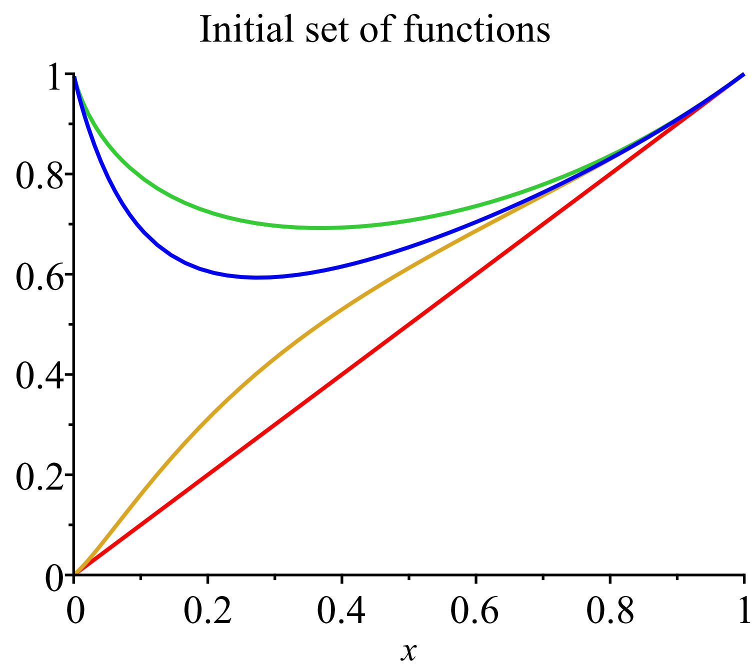
$$f := \left[1.732050808\,x, 2.730847567\,x^x - 3.301876898\,x, 19.66589123\,x^{x^x} - 18.39311402\,x - 2.515766748\,x^x, 59.60651264\,x^{x^{x^x}} - 37.86700971\,x - 55.21402071\,x^x + 32.45868473\,x^{x^x}\right] \quad (6.14)$$

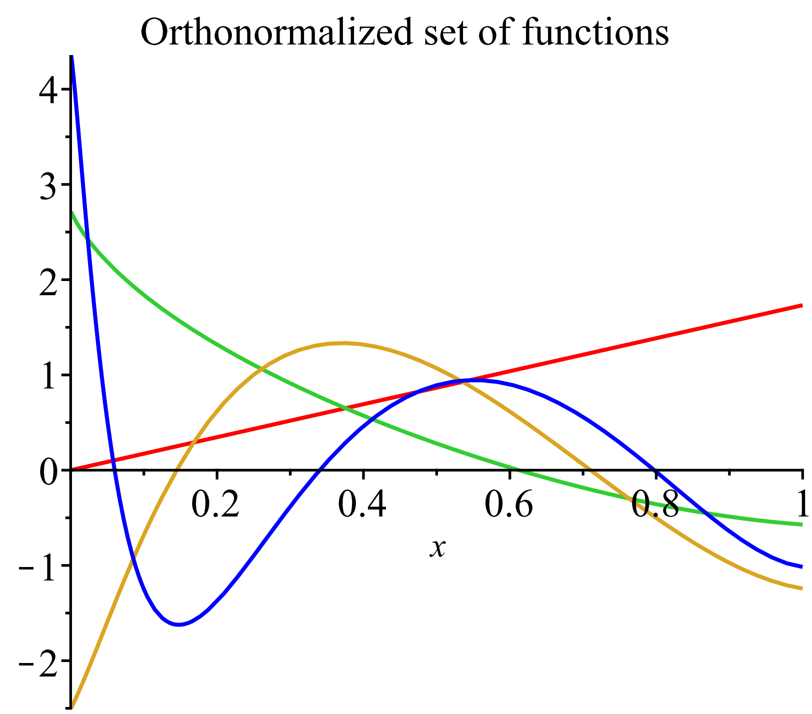
Orthonormality checking

> $evalf\left(\int_0^1 f_1^2\,dx\right), evalf\left(\int_0^1 f_2^2\,dx\right), evalf\left(\int_0^1 f_3^2\,dx\right), evalf\left(\int_0^1 f_4^2\,dx\right);$
$evalf\left(\int_0^1 f_1 f_2\,dx\right), evalf\left(\int_0^1 f_1 f_3\,dx\right), evalf\left(\int_0^1 f_1 f_4\,dx\right), evalf\left(\int_0^1 f_2 f_3\,dx\right), evalf\left(\int_0^1 f_2 f_4\,dx\right),$
$evalf\left(\int_0^1 f_3 f_4\,dx\right)$

$$1.000000000, 0.9999999999, 1.000000001, 1.000000000$$
$$-1.214117181\;10^{-9}, -7.262032476\;10^{-9}, -2.114557211\;10^{-8}, 1.519864918\;10^{-8}, 7.239409612\;10^{-8}, 6.520635115\;10^{-9} \quad (6.15)$$

```
> plot([op(F)], x=0..1, title="Initial set of functions");
  plot([op(f)], x=0..1, title="Orthonormalized set of functions");
```





Numeric orthonormalization of multivariate functions

> $F := [x \cdot y, \ln(1 + x\sin(y)), \ln(1 + x^2 \sin(2\,y)), \ln(1 + x^3 \sin(3\,y))]$

$$F := [x\,y, \ln(1 + x\sin(y)), \ln(1 + x^2 \sin(2\,y)), \ln(1 + x^3 \sin(3\,y))] \qquad \textbf{(6.16)}$$

> $f := GramSchmidtL2(F, [x = 0\,..1, y = 0\,..1], output = numeric)$

$$f := [3.000000000\,x\,y, 40.51210393\ln(1. + x\sin(y)) - 29.86666401\,x\,y, 13.43379645\ln(1. + x^2\sin(2.\,y)) + 1.901286700\,x\,y - 16.65023106\ln(1. + x\sin(y)), 40.04984137\ln(1. + x^3\sin(3.\,y)) + 53.68252999\,x\,y \qquad \textbf{(6.17)}$$

$- 37.06949338 \ln(1. + x \sin(y)) - 60.61251132 \ln(1. + x^2 \sin(2. y))]$

Orthonormality checking

$> evalf\left(\int_0^1\int_0^1 f_1^2 \,dx\,dy\right), evalf\left(\int_0^1\int_0^1 f_2^2 \,dx\,dy\right), evalf\left(\int_0^1\int_0^1 f_3^2 \,dx\,dy\right), evalf\left(\int_0^1\int_0^1 f_4^2 \,dx\,dy\right);$

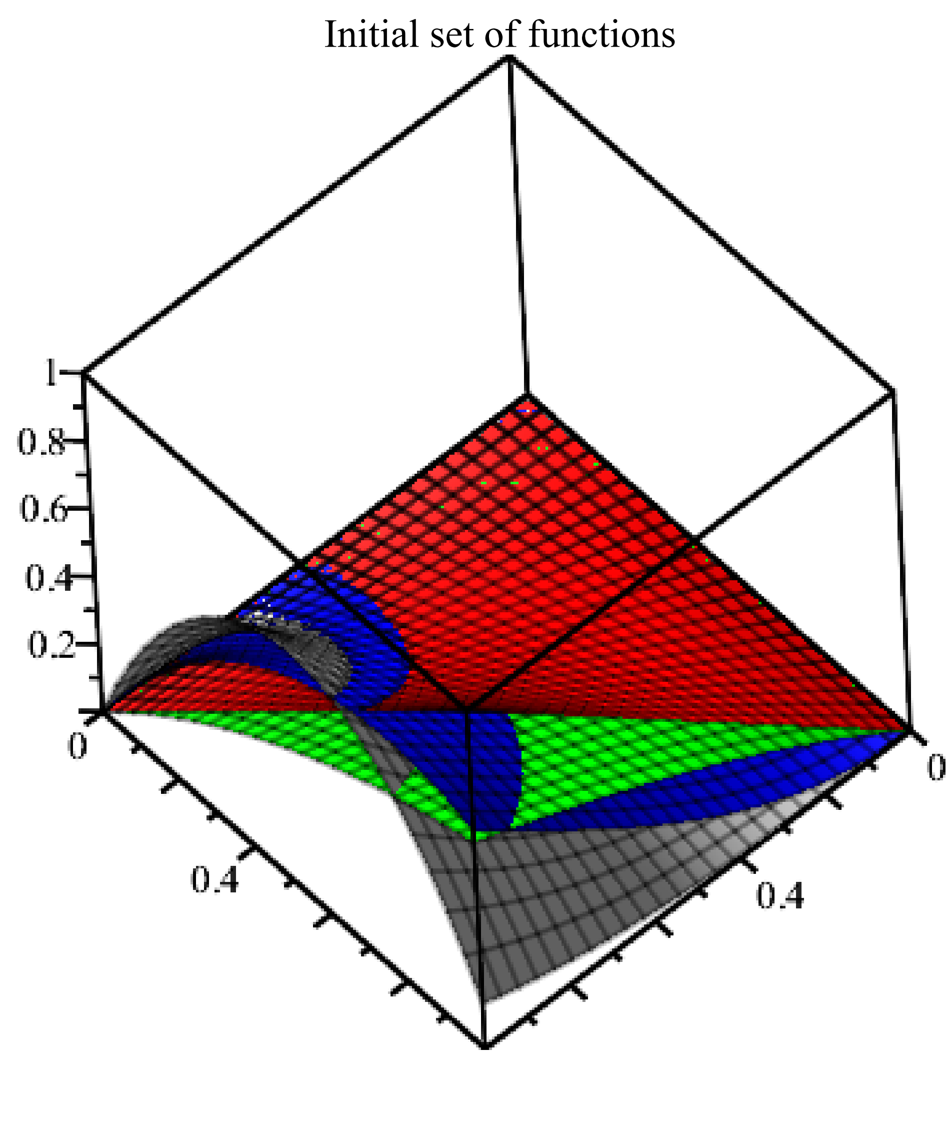
$evalf\left(\int_0^1\int_0^1 f_1 f_2 \,dx\,dy\right), evalf\left(\int_0^1\int_0^1 f_1 f_3 \,dx\,dy\right), evalf\left(\int_0^1\int_0^1 f_1 f_4 \,dx\,dy\right), evalf\left(\int_0^1\int_0^1 f_2 f_3 \,dx\,dy\right), evalf\left(\int_0^1\int_0^1 f_2 f_4 \,dx\,dy\right), evalf\left(\int_0^1\int_0^1 f_3 f_4 \,dx\,dy\right)$

$1.000000000, 1.000000000, 1.000000000, 1.000000000$

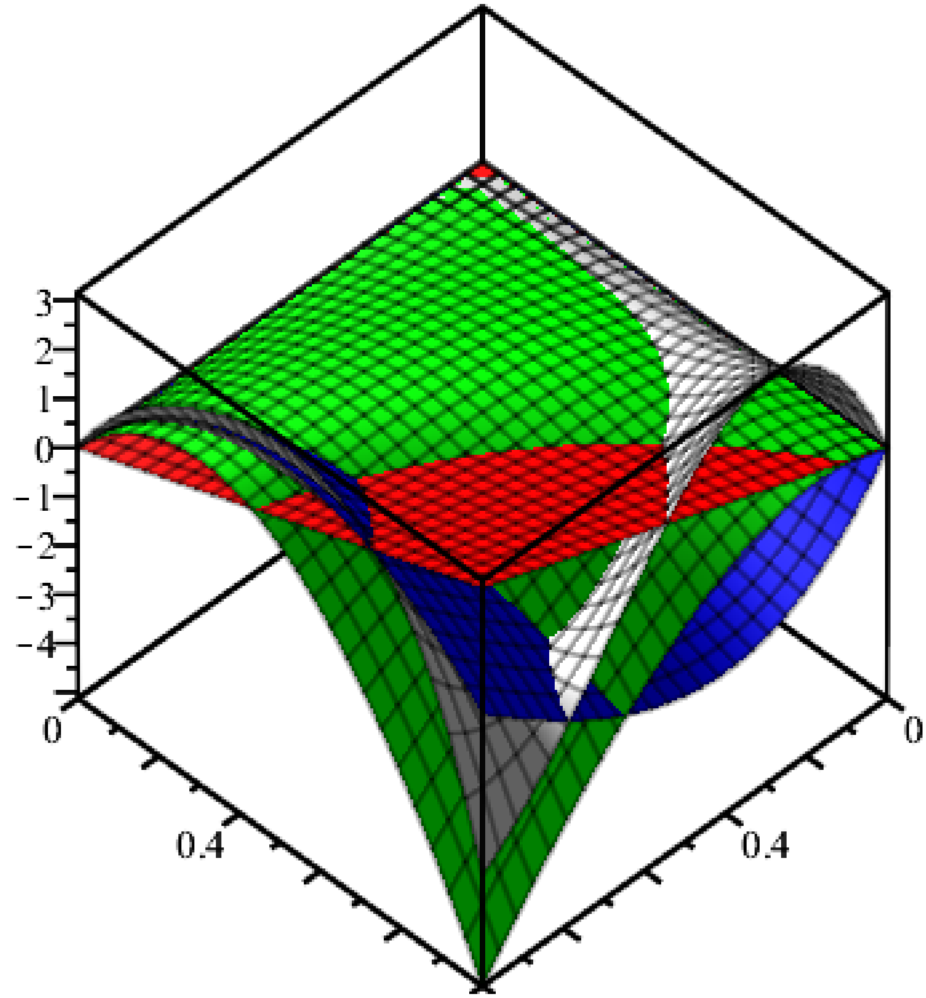
$-1.016745715\ 10^{-9}, -1.697662497\ 10^{-9}, 1.942032160\ 10^{-9}, 3.201623628\ 10^{-9},$ **(6.18)**

$-8.979597590\ 10^{-9}, 2.120226923\ 10^{-9}$

> *plot3d*([*op*(*F*)], *x* = 0 ..1, *y* = 0 ..1, *color* = [*red*, *green*, *blue*, *gray*], *title* = "Initial set of functions", *plotlist*);
plot3d([*op*(*f*)], *x* = 0 ..1, *y* = 0 ..1, *color* = [*red*, *green*, *blue*, *gray*], *title* = "Orthonormalized set of functions", *plotlist*);



Orthonormalized set of functions



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