Classroom Tips and Techniques: Plotting a "Slice" of a Vector Field

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Introduction

Not long ago, a user contacted me with the question "Can Maple plot a slice of a vector field, and can it superimpose this slice on a density plot?"   The short answer is "yes," and the long answer hinges on the meaning of "slice of a vector field." However, after thinking about what this might mean, it suddenly came clear that what the user wanted was a graph of the arrows of a field emanating from a given surface.   In particular, the surface was a coordinate plane in Cartesian space, but Maple has the ability to draw the field arrows emanating from an arbitrary surface.

In this month's article we'll give an example to show how both the desired graphs can be obtained.

Initializations

restart
with(*VectorCalculus*) :
with(*plots*) :
with(*plottools*) :

The Vector Field

A vector field in Maple is ordinarily created with the **VectorField** command, typically in the form shown in Table 1.

$VectorField([x, y, z], cartesian[x, y, z])$ $$(x)\bar{e}_x + (y)\bar{e}_y + (z)\bar{e}_z$$
Table 1   Typical usage of the **VectorField** command

The display format by default contains the unit basis vectors explicitly. The symbols $\bar{e}_x, \bar{e}_x, \bar{e}_x$ are the vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ of a standard calculus course. Table 2 shows the display of this field as a column vector, without the explicit use of basis vectors.

BasisFormat(*false*) :
VectorField ([x, y, z], *cartesian* [x, y, z])
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$
Table 2   A vector field displayed without basis vectors

By default, the **BasisFormat** command is set to *true*, but we will leave it set to *false*.

The vector field in the user's request arose as the gradient of the scalar field

$$f(x, y, z) := \sqrt{x^2 + y^2 + z^2}$$

In Maple's *VectorCalculus* package, the **Gradient** command produces a vector field. Table 3 contains the vector field of interest.

F := *Gradient* ($f(x, y, z)$, [x, y, z])
$$\begin{bmatrix} \dfrac{x}{\sqrt{x^2 + y^2 + z^2}} \\ \dfrac{y}{\sqrt{x^2 + y^2 + z^2}} \\ \dfrac{z}{\sqrt{x^2 + y^2 + z^2}} \end{bmatrix}$$
Table 3   The gradient vector field

A "Slice" of the Vector Field

The "slice" of the vector field is a graph of a coordinate plane from which the arrows of the field emanate. Fortunately, the **PlotPositionVector** command in the *VectorCalculus* package is exactly the right command. It is designed to represent either curves or surfaces in radius-vector,

or position-vector form.   As such, it is always a vector emanating from the origin of a Cartesian space.   Table 4 shows the position vector that represents the plane curve
$x(t) = 2 \cos(t), y(t) = 3 \sin(t)$.

$\mathbf{C} := PositionVector([2 \cos(t), 3 \sin(t)], cartesian[x, y])$

$$\begin{bmatrix} 2\cos(t) \\ 3\sin(t) \end{bmatrix}$$

Table 4   The position vector representation of the plane curve $x(t) = 2 \cos(t), y(t) = 3 \sin(t)$

This vector representation of the ellipse can be plotted with the **PlotPositionVector** command. Moreover, a variety of vector fields along this curve can easily be added by invoking appropriate options in the command.   Thus, the tangent and normal fields, various derivative fields and even arbitrary vector fields along the curve can be graphed. Figure 1 shows the ellipse in Table 4, along with the tangent field.
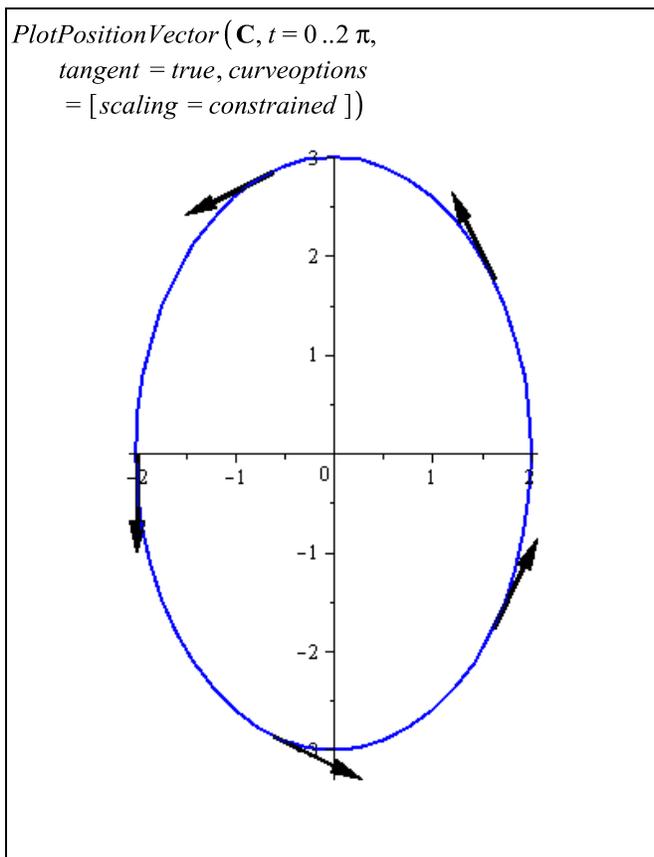
$PlotPositionVector(\mathbf{C}, t = 0..2\,\pi, tangent = true, curveoptions = [scaling = constrained])$

Figure 1   The ellipse of Table 4 along with the tangent field

Of course, the number, location, and properties of the tangent vectors can be controlled. Although the graph in Figure 1 can be painstakingly constructed by joining graphs of the curve and tangent vectors located precisely along it, the utility of the **PlotPositionVector** command should be obvious.

Surfaces can also be represented with a position vector.   Table 5 provides the position-vector representation of the coordinate plane $x = 1$.

$\mathbf{R}_1 := PositionVector([1, u, v],$
$\quad cartesian[x, y, z])$

$$\begin{bmatrix} 1 \\ u \\ v \end{bmatrix}$$

Table 5   Position-vector representation of the coordinate plane $x = 1$

Figure 2 shows the arrows of the gradient field **F** emanating from the coordinate plane $x = 1$. This is the "slice" of the vector field I was asked to produce.
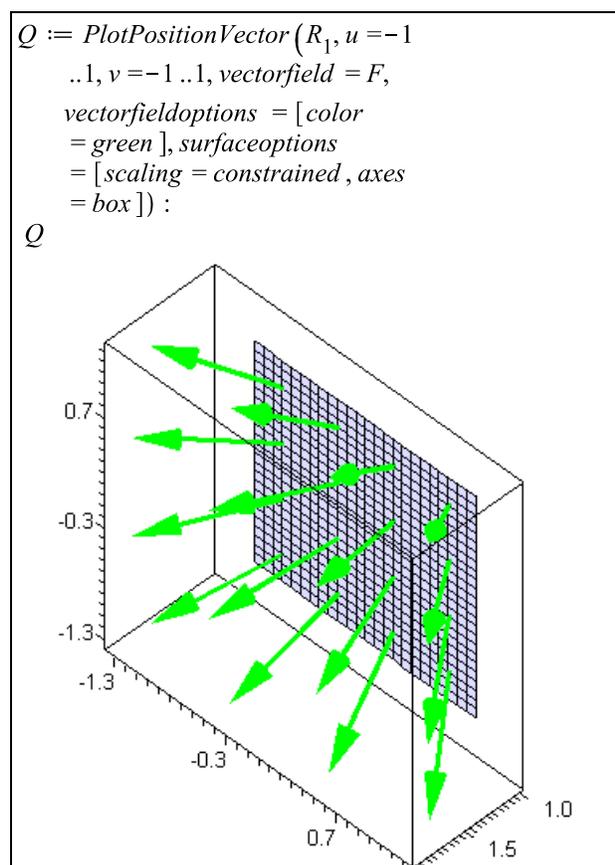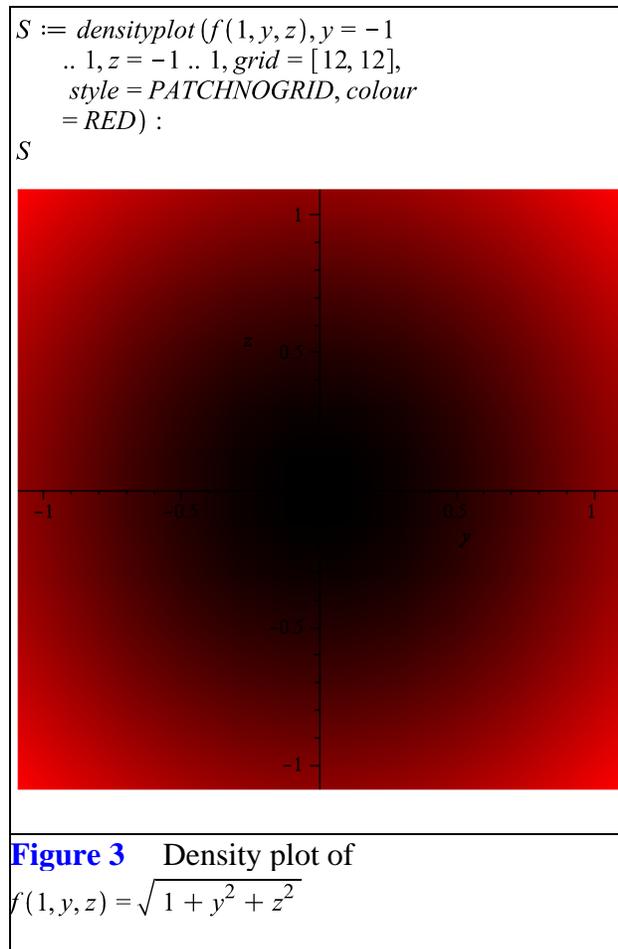
$Q := PlotPositionVector(R_1, u = -1$
$\quad ..1, v = -1..1, vectorfield = F,$
$\quad vectorfieldoptions = [color$
$\quad = green], surfaceoptions$
$\quad = [scaling = constrained, axes$
$\quad = box]):$
Q

The slice of the gradient field shown in Figure 2 has been assigned to the variable Q so that in the next section we can superimpose this image on a density plot of the scalar field whose gradient arrows we see in the figure.

Density Plot of the Scalar Field

Figure 3 is a density plot of $f(1, y, z)$.   It gives a measure of the value of $f(1, y, z)$.

$S := densityplot\,(f(1, y, z), y = -1$
$\quad .. \ 1, z = -1 .. \ 1, grid = [12, 12],$
$\quad\quad style = PATCHNOGRID, colour$
$\quad\quad = RED) :$
S

Figure 3   Density plot of
$f(1, y, z) = \sqrt{1 + y^2 + z^2}$

Note that again we have assign a plot structure to a name, here, S.   The associated plot represents as a density, the scalar field evaluated on the coordinate plane $x = 1$.   Our challenge now is to superimpose this image on the coordinate plane $x = 1$ in Figure 2.   The challenge is inherent in the different dimensions for the two figures.   Figure 2 is a 3D graph, but Figure 3 is
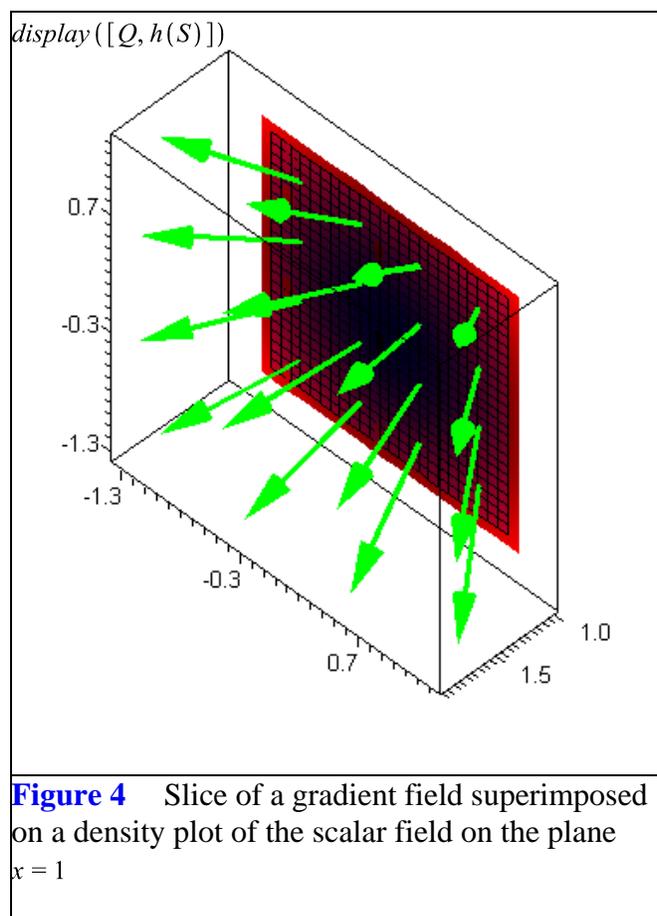
only a 2D graph.   As such, they cannot be combined.

Merging Plots of Different Dimensions

To merge Figures 2 and 3, we need to convert Figure 3, a 2D plot, into a 3D plot.   We do this by adding a third coordinate to each data point in the plot data structure on which Figure 3 is based. Thus, each 2D point (y, z) must become the 3D point $(1, y, z)$.   This is done with the **transform** command from the *plottools* package, with which we define the function

$$h := transform\,((y, z) \rightarrow [1, y, z]) :$$

Now, the **display** command from the *plots* package can be invoked to merge Figure 2 and the transformed Figure 3.   The result is the final image in Figure 4.

Figure 4   Slice of a gradient field superimposed on a density plot of the scalar field on the plane $x = 1$