Minimal Road Radius for Highways

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This problem solves for the minimal radius of curvature for designing and building a banked curve on a road assuming a constant speed and elevation. It is written using the approach found in the modern introductory undergraduate physics textbook: *Matter and Interaction*, by Chabay and Sherwood. It demonstrates the pedagogical value of Maple's ability to teach physics and to solve problems starting from fundamental principles, i.e., a top-down approach. This is in contrast to most computational systems where one codes starting from a specific example before implementing the fundamental principles.

This application can be used to:

a) confirm the values found in the manual for the American Association of State Highway and Transportation Officials (AASHTO) that engineers use to design and build these banked curves are physically sound.

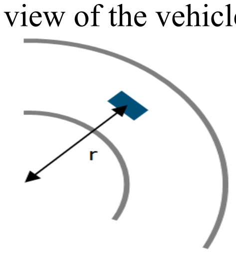
b) highlight the pedagogical value inherent in the Maple language to distinguish between assignment (:=) and equivalence (=);

c) most importantly, to demonstrate the pedagogical value Maple has in thinking about solving a problem involving a physical process. Given Maple's symbolic mathematics capabilities, one can implement a top-down approach to the physics and the mathematics, working from the general principle to the specific example. This allows one to avoid the types of errors that occur when translating the problem into a bottom up approach, from specific values of the example to the general principle, an approach that is required by most other computational systems.

I hope that others are willing to continue to engage in discussions related to the pedagogical value of Maple beyond mathematics.

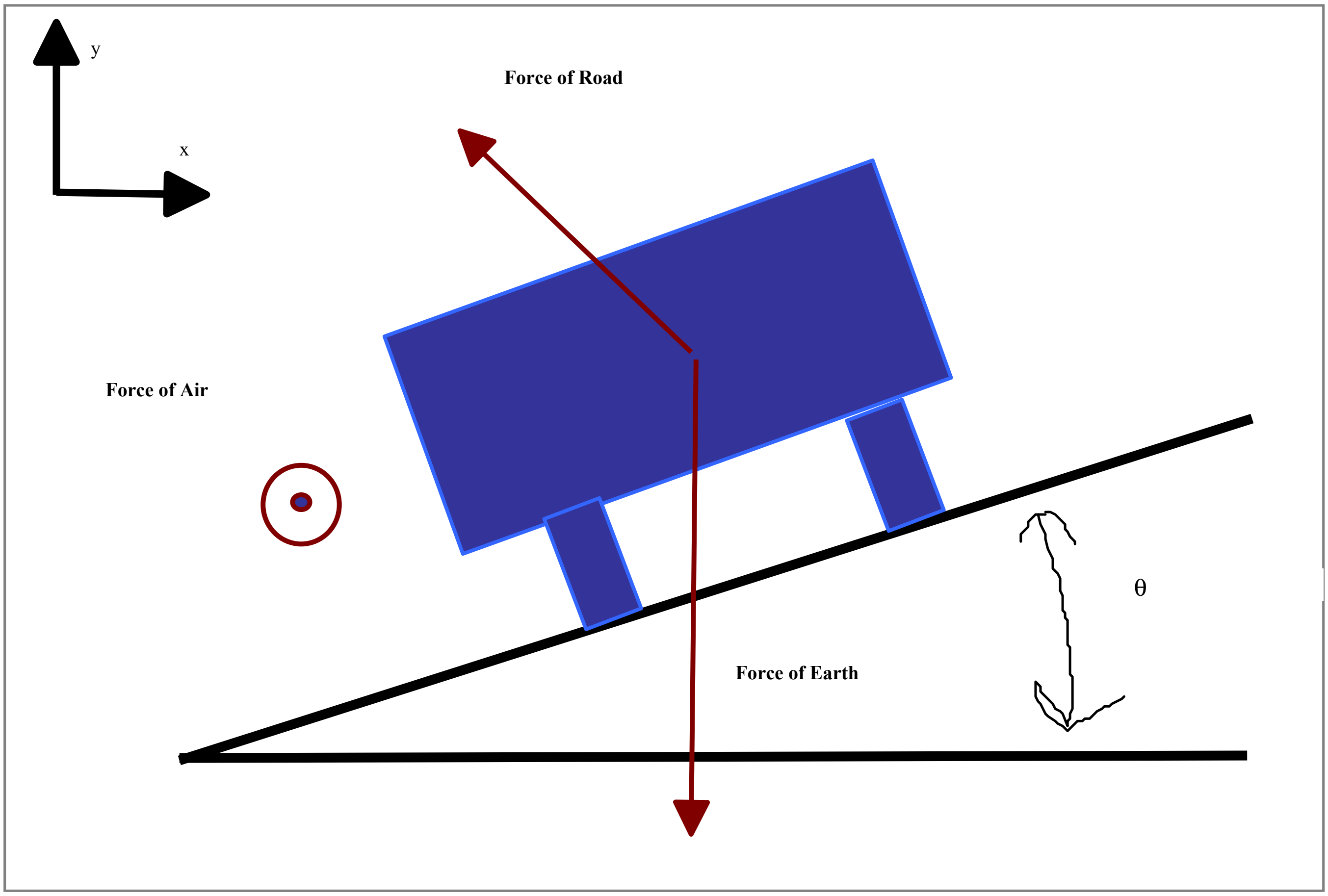
The problem:

The problem is to determine the minimal radius of curvature of a banked highway curve for a vehicle traveling at a fixed speed and altitude. This radius is determined by the following parameters: the speed of the vehicle through the curve, the gradient of the banking, and frictional component between the tires of the vehicle and the road. The top-down view of the vehicle traveling through the curve is shown here:



While in the turn, the vehicle is to maintain a constant speed and its vertical elevation does not change. The curve has a radius of curvature of *r*. Notice *r* is purely horizontal. We will analyze the interactions the vehicle has with the road using this next figure.

In this cross-sectional view of its motion, the vehicle is traveling *away* from the reader. Note the coordinate system chosen in the figure.



Since this is a mechanical system, we will analyze this situation using the Momentum Principle, (often referred to as Newton's 2nd Law of motion in traditional physics textbooks):

$$\frac{d}{dt}\vec{p} = \vec{F}_{net}$$

where $\vec{p}$ is the vector description of the momentum of a body, and $\vec{F}_{net}$ is the vector sum of the external forces on the body. In this system, the body will be the vehicle. For convenience, and because there is no way the vehicle is traveling at a speed close to the speed of light: $\vec{p} \approx m\,\vec{v}$ where m is the mass of the vehicle, and $\vec{v}$ is the velocity of the vehicle.

Even though the vehicle travels with a constant speed and elevation, because it is changing its direction, it's velocity, and thus momentum, is changing Hence, we know there is a non-zero net force acting on the vehicle. If there were no net force, the vehicle would continue to travel in a straight line.

Let us create a vector-based equation which we will name: $\overrightarrow{p\_eq}$. (The vector arrow above the term is to remind us that this a vector based equation.)   To this name, we will **assign** the momentum principle equation. The momentum principle equation describes the **equivalence** of the time rate of change of momentum to the net force:

> $restart;$
>
> $\overrightarrow{p\_eq} := \overrightarrow{dpdt} = \vec{F}_{net}:$

We will use the cross-sectional view figure to write out all the vectors. In Cartesian coordinates, the $+x$ direction is to the right, the $+y$ direction is up, the $+z$ direction is out of the page. In this figure, the direction of the change of momentum is completely to the left. Hence we can assign components to the $\overrightarrow{dpdt}$ vector. Because only the direction of the magnitude of the momentum is changing, we can show geometrically that the magnitude of the time rate of change in momentum is equivalent to $\frac{m\,v^2}{r}$. The vehicle is moving in a circle in the first figure, or what some people refer to as centripetal motion For the vector $\overrightarrow{dpdt}$ we use the coordinates of the second figure and assign to it the following components in symbolic form:

> $\overrightarrow{dpdt} := \left\langle -\frac{m\,v^2}{r}, 0, 0 \right\rangle :$

Note the negative sign in the x component. This is because the direction of change is to the left.

Next we work on the right-hand-side of the momentum principle equation. The net force is due to the sum of all the forces caused by all the interactions of the vehicle with the environment, i.e., that which is not in the system. There are three significant entities:
* the air molecules, which are hitting the vehicle in the front as it move into the $-z$ direction,
* the Earth, which is interacting with the vehicle through its gravitational field,
* the road, which is making solid contact with the tires of the vehicle.

Hence we can assign to the net force term the vector addition of theses forces of interaction:

> $\vec{F}_{net} := \vec{F}_{air} + \vec{F}_{Earth} + \vec{F}_{road}:$

Now we discuss each interaction force.

Air: The air molecules collided with the vehicle in the opposite direction that the vehicle is traveling. That direction in the second figure is purely in the $+z$ direction. Hence we can assign to the vector $\vec{F}_{air}$ its components:

> $\vec{F}_{air} := \langle 0, 0, +F_{air} \rangle :$

where F_{air} is the magnitude of the force of the air molecules.

The Earth: it is interacting with the vehicle through the gravitational field of the Earth. At the surface, that field is pointing purely in the $-y$ direction. Hence we can write:

> $\vec{F}_{Earth} := m \cdot \vec{g} :$
>
> $\vec{g} := \langle 0, -g, 0 \rangle :$

Note the difference between the vector field expression, $\vec{g}$ and its magnitude, g.

The road: the force of the road where the wheels are making contact with the road. The force vector of a solid-solid interaction is usually broken up into two vectors. One force vector of the road is that of it pushing directly against the vehicle compressing the atoms in the tires. The other force vector is that of the road pushing on the wheels *sideways* to the interface to keep the car wheels from slipping with respect to the road. This anti-slipping force is what we often refer to as the friction of the road. And since we don't want the tires to slip, this is referred to as *static* friction.

We can assign these classifications of forces to the force of the road vector:

$$> \vec{F}_{road} := \vec{F}_{compression} + \vec{F}_{friction} :$$

The road compressing the tires, which is the force of reciprocity of the tires compressing the, (also known as the reaction force in Newton's 3rd law), is perpendicular or "normal" to the interface of the two solids. Traditionally this is referred to as the "normal force", but often, this term leads to much confusion by students and some instructors. There is no force of a "perpendicular"; there is a force of the road. Let us calculate the compression force. Since the interface along a line at an angle θ to the *x* axis on the x-y plane, the perpendicular direction at an angle $\theta + \frac{\pi}{2}$ from the +*x* direction. As always, the *x*-component of the vector is related to the magnitude of the vector and the cosine of the angle, while the *y*-component of the vector is related to the magnitude of the vector and the sine of the angle. Hence we assign to this compression force its components:

$$> \vec{F}_{compression} := \left\langle F_c \cdot \cos\left(\theta + \frac{\pi}{2}\right), F_c \cdot \sin\left(\theta + \frac{\pi}{2}\right), 0 \right\rangle :$$

F_c is the magnitude of the compression force. Now, you may have learned how to trigonometrically simplify these components, but why apply them. Maple will do it all for you! So set it up correctly and let Maple do the math.

The friction between the vehicle wheels and the road is parallel to the interface, hence it can be broken up into two vectors, one which is in the same direction the the direction of motion, and one which is perpendicular to the direction of motion. If the road does not provide a component parallel to the direction of the motion, the air will slow down the vehicle. This force of propulsion by the road is the force of reciprocity, or reaction force, of the tires pushing on the road to move the car forward. Hence this vector component of the friction by the road is purely in the -*z* direction.

The other vector force of the friction term is the one that which is keeping the vehicle from slipping perpendicular to its direction of motion, either up or down the banked curve. It is purely in *x-y* plane of the figure above. Since the vehicle is moving fast and the curve is turning to the left, the friction will keep the car from sliding up over the top of the banked curve. Hence its direction is also to the left, but parallel to the interface, hence down the embankment. [Aside: Notice in the second figure the force of the road is not purely perpendicular to the road-tire interface? That because it is a combination of this frictional force and the compression force.] By the way, if one guesses wrong here about the direction of the force, the resulting radius calculations will be negative.

Given the force that is pointing down the road toward the center, the angle of this frictional component vector to the +*x* direction is, in radians, $\theta + \pi$. Thus we can now make the following 3 assignments:

> $\vec{F}_{friction} := \vec{F}_{propulsion} + \vec{F}_{slip}$:
> $\vec{F}_{propulsion} := \langle 0, 0, -F_{propulsion} \rangle$:
> $\vec{F}_{slip} := \langle F_f \cdot \cos(\theta + \pi),\ F_f \cdot \sin(\theta + \pi), 0 \rangle$:

F_f is the magnitude of this no-slipping frictional force.

One last bit of physics:
* we know that experimentally, we can approximate the magnitude of the component of the friction to keep the vehicle from slipping with the magnitude of the compression force:

$$\left|\vec{F}_{slip}\right| \leq \mu \cdot \left|\vec{F}_{compression}\right|.$$

The term μ is called the coefficient of *static* friction. Since we want to find the minimal radius, thus we need to apply the maximum friction, we can use the full expression for the magnitude of the slipping frictional force. Hence we can assign to the magnitude of the frictional force vector:

> $F_f := \mu \cdot F_c$:

We seem to have a great deal of equations. But so what! **Let Maple do the math!** By starting with a top down approach, and filling each term with the combination of more detailed terms, we are unlikely to make an error in the application of the physics and the mathematical relationships.

Now that we have continually added more details to our momentum principle relationship, it is the time to see what the highly detailed vector equation looks like: (*Note*, the Maple command *simplify* is not required for this example, but is good practice. *simplify* forces Maple to evaluate every term when it shows it to us.)

> $simplify(\overrightarrow{p\_eq})$;

$$\begin{bmatrix} -\frac{m v^2}{r} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -F_c\left(\cos(\theta)\,\mu + \sin(\theta)\right) \\ -m g + F_c \cos(\theta) - \mu F_c \sin(\theta) \\ F_{air} - F_{propulsion} \end{bmatrix} \qquad \textbf{(1)}$$

A reminder, our goal is to determine the relationship of *r*, the radius of curvature of the turn, to all the known properties of the system.

What are the other unknowns in this vector equation? We don't know the force of the air, nor the force of the ground propelling the vehicle forward, but they aren't important in the calculation the *r* so we will ignore the z-component of the equation. (Even without Maple we can see that $F_{air} = F_{propulsion}$ and they disappear from the goal of this problem.) We don't know the force of compression of the road back onto the vehicle, but we don't know the mass of the vehicle. Maybe the mass will cancel itself out since we don't build different curves for vehicles of different weights. It is assumed we know the speed of the vehicle and the coefficient of friction.

Here it would be great if we could simply give Maple the vector equation and ask it to solve for the scalar values. However, the command *solve* requires a list, or set, of equations. Hence we have to break our vector equation up into an equation for each component. For me, the most readable way to see this is to equate each component of the left hand side of the equation to its equivalent component of the right hand side and assign these *equivalences* to the name *p_eqs*:

> $p\_eqs := seq(lhs(\overrightarrow{p\_eq})[n] = rhs(\overrightarrow{p\_eq})[n], n = 1..2);$

$$p\_eqs := -\frac{m\,v^2}{r} = -F_c \sin(\theta) - \mu F_c \cos(\theta),\ 0 = -m\,g + F_c \cos(\theta) - \mu F_c \sin(\theta) \quad \textbf{(2)}$$

Note we are using only the first 2 equations / components. Now we can solve the set "{ }" of equations for the set of unknown variables:

> $solutions := solve(\{p\_eqs\}, \{F_c, r\});$

$$solutions := \left\{F_c = \frac{m\,g}{\cos(\theta) - \mu \sin(\theta)},\ r = \frac{v^2\,(\cos(\theta) - \mu \sin(\theta))}{g\,(\cos(\theta)\,\mu + \sin(\theta))}\right\} \quad \textbf{(3)}$$

This allows us to solve for the radius of curvature, r :

$$r = \frac{v^2\,(\cos(\theta) - \mu \sin(\theta))}{g\,(\mu \cos(\theta) + \sin(\theta))}$$

It shows that the radius is independent of the mass of the vehicle, which is great since we don't have to build different shaped turns for different vehicles. Actually, it is not a surprised because of the equivalence of the inertial mass to the gravitational mass. But that is for another discussion.

Now we ask: does the expression for r make sense? Let us vary the variables in the solution and see if we obtain physically reasonable behavior.

1) The faster the vehicle, the greater the radius one has to build the turn. Yes, that makes sense in our own sense of trying to change directions while running vs. walking on a flat surface. It is easier to turn the corner quickly when walking than when running.

2) The cos(θ) term in the numerator shows how banking the curve more, i.e., increasing the angle, the smaller the radius that is required. This makes physical sense as the heavily banked road can help us turn the corner more quickly.

3) The -μ sin(θ) term in the numerator means that the friction of the road can used to reduce the radius. The greater the friction, μ, the tighter the radius can be.

So it looks good. Remember, if we are working with a slower vehicle for which the friction keeps the vehicle from slipping *down* the bank, then we must return to the calculation, and reset the direction of the slip avoidance friction force. (We leave this as an exercise to the reader. **But the great thing about Maple is that one just needs to change one line, re-execute the worksheet (!!!) and the answer appears instantaneously**.)

While in the field, it likely more convenient for engineers to measure the radius curvature from the center of the curve to where the car is traveling on the elevated bank which is at a higher altitude. This means r is only the horizontal component of this measurement: $r = R\cos(\theta)$.

Let us extract the relationship of *r,* divide by the cos of the angle, and assign it to a variable: *R*.

> $R := \frac{eval(r, solutions)}{\cos(\theta)};$

$$R := \frac{v^2 \left(\cos(\theta) - \mu \sin(\theta)\right)}{g \left(\cos(\theta)\, \mu + \sin(\theta)\right) \cos(\theta)} \qquad \textbf{(4)}$$

Examples:
Now let's apply some examples. For pedagogical reasons, I prefer if students work exclusively with SI units and let Google, or a function, calculate the MKS value.

I'll write a function to convert from kph to meters-per-second which is the SI unit for speed.

> $kph2mps(v) := v \cdot 1000 \cdot \frac{1}{60} \cdot \frac{1}{60}:$

Example 1: speed is 30 [kph], with a coefficient of friction of 0.28. The bank is at a gradient of 6, which means $\theta = \arctan\left(\frac{6}{100}\right)$. We know g is 9.8 [m / s^2].

> $interface(displayprecision = 2):$
$myR := eval\left(R, \left\{v = kph2mps(30.), g = 9.8, \mu = 0.28, \theta = \arctan\left(\frac{6.}{100}\right)\right\}\right)$

$$myR := 20.53 \qquad \textbf{(5)}$$

So, the distance from the center up to where the vehicle travels around the curve is 20.5 [m]. For fun, we can compare the value with the value printed in the manual for the American Association of State Highway and Transportation Officials (AASHTO) for minimal road radius. The value they post is 20.8 [m]. Slightly different, but not much. Let's see if our difference becomes larger as we examine faster speeds.

Example 2: speed is 110 [kph] which is again banked with a gradient of 6, but with a coefficient of friction of 0.11

> $myR := eval\left(R, \left\{v = kph2mps(110), g = 9.8, \mu = 0.11, \theta = \arctan\left(\frac{6.}{100}\right)\right\}\right);$

$$myR := 557.71 \qquad \textbf{(6)}$$

or 558 [m] which is comparable to AASHTO's value of 560.4 [m]. The relative difference is quite small and is probably due to the engineers constructing the manual rounding intermediate steps in the process of calculating R. There is a very small relative percentage difference between our calculations and their value. It is:

> $AASHTO\_R := 560.4:$

$reldiffpct := \frac{(AASHTO\_R - myR)}{myR} \cdot 100$

$$reldiffpct := 0.48 \qquad \textbf{(7)}$$

or about an additional one-half percent. Very little, and they are leaning toward the safer side of this calculation. Good move.

If you want to use Maple's units conversion, here is an example. Make sure you load the Units package/library. Then use the Units palette. I do admit that it is cool that one can mix and match units in working problems. You can even change the units of the output to something other than default SI units. (My only beef about Maple Units is that I wish they had retained the double brackets format around the units. With the units letters such as "m", one can read it multiple ways: as the unit meter or as an unevaluated variable.)

Example 3: AASHTO includes the speed of 80 mph (Wow! Must be for highways in Montana.) Using a bank with a gradient of 6 and a minimal friction of 0.08, we calculate:

> $with(Units{:}\text{-}Simple):$

> $eval\left(R, \left\{v = 80\ \text{mph}, g = 9.8\ \text{m}/\text{s}^2, \mu = 0.08, \theta = \arctan\left(\frac{6.}{100}\right)\right\}\right)$

$$3049.26\ \text{ft} \tag{8}$$

or about 3049 [ft]. The AASHTO manual states 3047.6 [ft], but says the "rounded radius (ft)" value is: 3050. In short, and fortunately for all of us, their values appear to physically sound given the error that often occurs in measuring both the coefficient of friction and the gradient.