Gödel's Universe

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In 1949, Kurt Gödel proposed a solution to Einstein's field equations that exhibited a rotation of matter (with a nonvanishing cosmological constant). This solution allows the existence of closed timelike curves, which implies the possibility of time travel. In this worksheet, we demonstrate the use of **DifferentialGeometry** and **Tensor** packages to calculate the Ricci tensor, Ricci scalar, and Einstein tensor. We also apply the calculus of variations to calculate the null geodesics (trajectories of light rays) in Gödel's universe. The notation is based on the presentation in the monograph of Stephen Hawking and George Ellis.

To honor Albert Einstein on the occasion of his 70th birthday, Kurt Gödel presented "An example of a new type of cosmological solution of Einstein's field equations of gravitation" (published in *Reviews of Modern Physics* in 1949). Gödel's model admits the existence of closed timelike curves, implying that it is possible to go on a journey into one's own past. Because Gödel's model yields no red shift for distant objects, it is considered unphysical and not discussed in most textbooks on general relativity. In the text *Large scale structure of space-time* by Stephen Hawking and George Ellis, there is a terse treatment of Gödel's universe, see below.

> **5.7 Gödel's universe**
>
> In 1949, Kurt Gödel published a paper (Gödel (1949)) which provided a considerable stimulus to investigation of exact solutions more complex than those examined so far. He gave an exact solution of Einstein's field equations in which the matter takes the form of a pressure-free perfect fluid ($T_{ab} = \rho u_a u_b$ where ρ is the matter density and u_a the normalized four-velocity vector). The manifold is R^4 and the metric can be given in the form
>
> $$ds^2 = -dt^2 + dx^2 - \tfrac{1}{2}\exp(2(\sqrt{2})\,\omega x)\,dy^2 + dz^2 - 2\exp((\sqrt{2})\,\omega x)\,dt\,dy,$$
>
> where $\omega > 0$ is a constant; the field equations are satisfied if $\mathbf{u} = \partial/\partial x^0$ (i.e. $u^a = \delta^a{}_0$) and $$4\pi\rho = \omega^2 = -\Lambda.$$
>
> The constant ω is in fact the magnitude of the vorticity of the flow vector u^a.

Source: Hawking and Ellis (Ref. 1), page 168.

We use **DifferentialGeomety** and **Tensor** packages to calculate the Ricci tensor, Ricci scalar, Einstein tensor, and energy-momentum tensor.

```
> restart:
> with(DifferentialGeometry): with(Tensor):
> Preferences("ShowFramePrompt", false);
```

true **(1)**

Set up the frame:

```
> DGsetup([t, x, y, z], M);
```

$$\textit{frame name: M} \quad (2)$$

Define the metric according to the convention of Hawking and Ellis:

```
> g := evalDG(-dt &t dt + dx &t dx - 1/2*exp(2*sqrt(2)*omega*x)*dy
  &t dy + dz &t dz - exp(sqrt(2)*omega*x)*dt &t dy - exp(sqrt(2)*
  omega*x)*dy &t dt);
```

$$g := -dt\,dt - e^{\sqrt{2}\,\omega x}\,dt\,dy + dx\,dx - e^{\sqrt{2}\,\omega x}\,dy\,dt - \frac{e^{2\sqrt{2}\,\omega x}\,dy}{2}\,dy + dz\,dz \quad (3)$$

Calculate the Ricci tensor, Ricci scalar, and Einstein tensor:

```
> RicciTensor(g);
```

$$2\,\omega^2\,dt\,dt + 2\,\omega^2\,e^{\sqrt{2}\,\omega x}\,dt\,dy + 2\,\omega^2\,e^{\sqrt{2}\,\omega x}\,dy\,dt + 2\,\omega^2\,e^{2\sqrt{2}\,\omega x}\,dy\,dy \quad (4)$$

```
> RicciScalar(g);
```

$$-2\,\omega^2 \quad (5)$$

```
> EinsteinTensor(g);
```

$$3\,\omega^2\,D\_t\,D\_t - 2\,e^{-\sqrt{2}\,\omega x}\,\omega^2\,D\_t\,D\_y + \omega^2\,D\_x\,D\_x - 2\,e^{-\sqrt{2}\,\omega x}\,\omega^2\,D\_y\,D\_t + 2\,\omega^2\,e^{-2\sqrt{2}\,\omega x}\,D\_y\,D\_y + \omega^2\,D\_z\,D\_z \quad (6)$$

Double check that Einstein tensor can be constructed from Ricci tensor and Ricci scalar:

```
> evalDG(RicciTensor(g) - 1/2*RicciScalar(g)*g);
```

$$\omega^2\,dt\,dt + \omega^2\,e^{\sqrt{2}\,\omega x}\,dt\,dy + \omega^2\,dx\,dx + \omega^2\,e^{\sqrt{2}\,\omega x}\,dy\,dt + \frac{3\,\omega^2\,e^{2\sqrt{2}\,\omega x}\,dy}{2}\,dy + \omega^2\,dz\,dz \quad (7)$$

```
> RaiseLowerIndices(InverseMetric(g), %, [1, 2]);
```

$$3\,\omega^2\,D\_t\,D\_t - 2\,e^{-\sqrt{2}\,\omega x}\,\omega^2\,D\_t\,D\_y + \omega^2\,D\_x\,D\_x - 2\,e^{-\sqrt{2}\,\omega x}\,\omega^2\,D\_y\,D\_t + 2\,\omega^2\,e^{-2\sqrt{2}\,\omega x}\,D\_y\,D\_y + \omega^2\,D\_z\,D\_z \quad (8)$$

Calculate the energy-momentum tensor:

```
> T := evalDG(rho*D_t &t D_t);
```

$$T := \rho\,D\_t\,D\_t \quad (9)$$

Left-hand side of Einstein's field equations, with a cosmological constant term:

```
> l := evalDG(RicciTensor(g) - 1/2*g*RicciScalar(g) + Lambda*g);
```

$$l := -\left(-\omega^2 + \Lambda\right)dt\,dt - e^{\sqrt{2}\,\omega x}\left(-\omega^2 + \Lambda\right)dt\,dy + \left(\omega^2 + \Lambda\right)dx\,dx - e^{\sqrt{2}\,\omega x}\left(-\omega^2 + \Lambda\right)dy\,dt - \frac{e^{2\sqrt{2}\,\omega x}\left(-3\,\omega^2 + \Lambda\right)dy}{2}\,dy + \left(\omega^2 + \Lambda\right)dz\,dz \quad (10)$$

Right-hand side of Einstein's field equations, after changing the contravariant tensor to covariant one:

```
> r := evalDG(8*Pi*RaiseLowerIndices(g, T, [1, 2]));
```

$$r := 8\,\pi\,\rho\,dt\,dt + 8\,\pi\,e^{\sqrt{2}\,\omega x}\,\rho\,dt\,dy + 8\,\pi\,e^{\sqrt{2}\,\omega x}\,\rho\,dy\,dt + 8\,\pi\,e^{2\sqrt{2}\,\omega x}\,\rho\,dy\,dy \quad (11)$$

The equations needed to be satisfied:

```
> epr1 := evalDG(l - r);
```

$$epr1 := -\left(8\,\pi\,\rho - \omega^2 + \Lambda\right)dt\,dt - e^{\sqrt{2}\,\omega x}\left(8\,\pi\,\rho - \omega^2 + \Lambda\right)dt\,dy + \left(\omega^2 + \Lambda\right)dx\,dx \quad (12)$$

$$- e^{\sqrt{2}\,\omega x} \left(8 \pi \rho - \omega^2 + \Lambda\right) dy\, dt - \frac{e^{2\sqrt{2}\,\omega x} \left(16 \pi \rho - 3\, \omega^2 + \Lambda\right) dy}{2}\, dy + \left(\omega^2 + \Lambda\right) dz\, dz$$

```
> eq1 := Hook([D_t, D_t], epr1) = 0;
```

$$eq1 := -8 \pi \rho + \omega^2 - \Lambda = 0 \quad (13)$$

```
> eq2 := Hook([D_x, D_x], epr1) = 0;
```

$$eq2 := \omega^2 + \Lambda = 0 \quad (14)$$

```
> eq3 := Hook([D_y, D_y], epr1) = 0;
```

$$eq3 := -\frac{e^{2\sqrt{2}\,\omega x} \left(16 \pi \rho - 3\, \omega^2 + \Lambda\right)}{2} = 0 \quad (15)$$

Solve for Lambda and rho:

```
> solve({eq1, eq2, eq3}, {Lambda, rho});
```

$$\left\{\Lambda = -\omega^2, \rho = \frac{\omega^2}{4 \pi}\right\} \quad (16)$$

We found that the sign of the cosmological constant negative, corresponding to a positive pressure. After a change of coordinates, the line element takes the form shown in the following, which is page 169 of Hawking and Ellis.

the metric $\mathbf{g}_1$ takes the form

$$ds_1^2 = 2\omega^{-2}(-dt'^2 + dr^2 - (\sinh^4 r - \sinh^2 r)\, d\phi^2 + 2(\sqrt{2}) \sinh^2 r\, d\phi\, dt),$$

where $-\infty < t < \infty$, $0 \leqslant r < \infty$, and $0 \leqslant \phi \leqslant 2\pi$, $\phi = 0$ being identified with $\phi = 2\pi$; the flow vector in these coordinates is $\mathbf{u} = (\omega/(\sqrt{2}))\, \partial/\partial t'$. This form exhibits the rotational symmetry of the solution about the axis $r = 0$. By a different choice of coordinates the axis could be chosen to lie on any flow line of the matter.

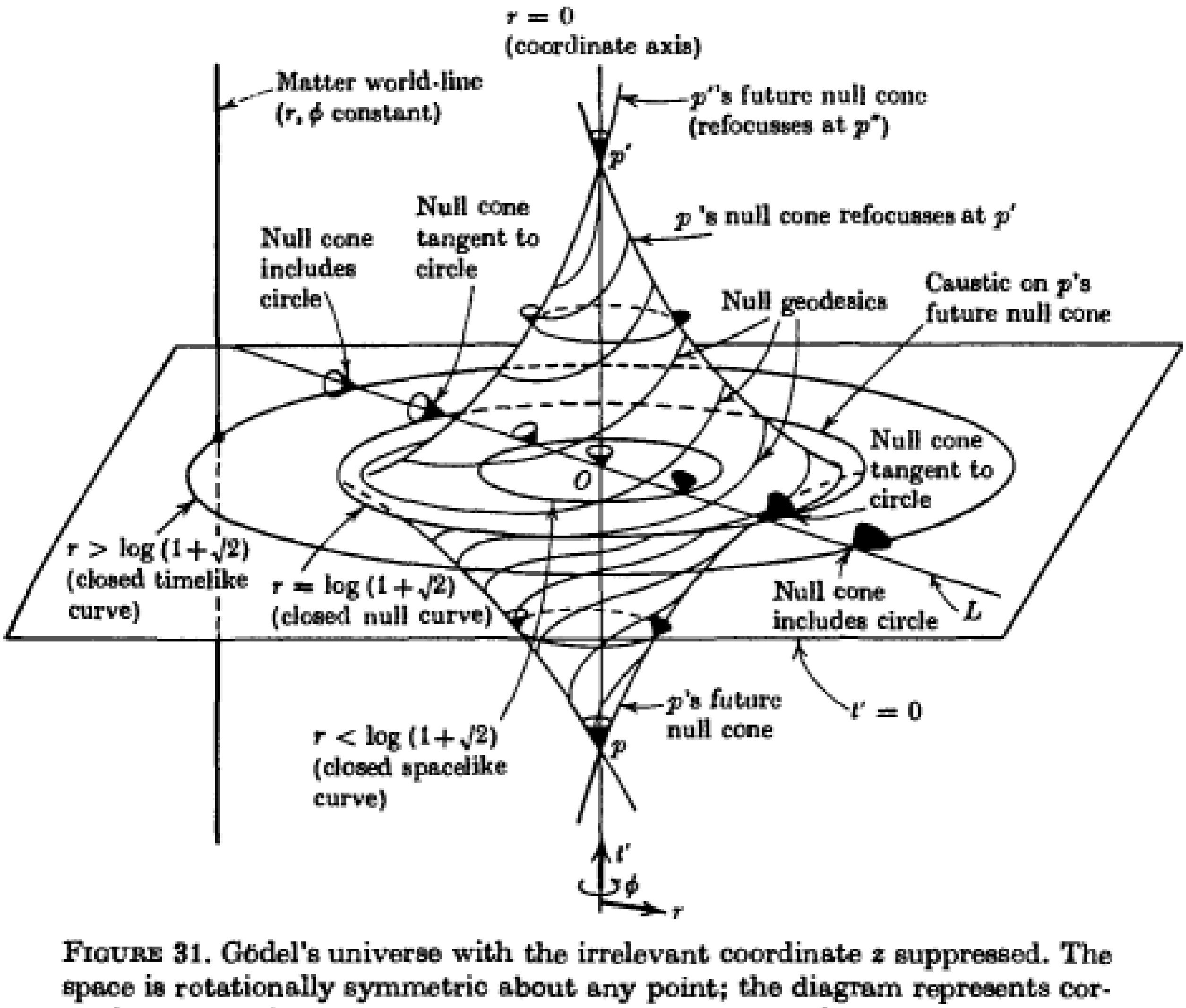


FIGURE 31. Gödel's universe with the irrelevant coordinate z suppressed. The space is rotationally symmetric about any point; the diagram represents correctly the rotational symmetry about the axis $r = 0$, and the time invariance. The light cone opens out and tips over as r increases (see line L) resulting in closed timelike curves. The diagram does not correctly represent the fact that *all* points are in fact equivalent.

Source: Hawking and Ellis (Ref. 1), page 169.

```
> restart:
```

Based on the line element, we write the Lagrangian, with lambda as the parameter:

```
> L := 1/omega^2*(-diff(t(lambda), lambda)^2 + diff(r(lambda), lambda)^2 - (sinh(r(lambda))^4 - sinh(r(lambda))^2)*diff(phi(lambda), lambda)^2 + 2*sqrt(2)*sinh(r(lambda))^2*diff(t(lambda), lambda)*diff(phi(lambda), lambda));
```

$$L := \frac{1}{\omega^2}\left(-\left(\frac{d}{d\lambda}\,t(\lambda)\right)^2 + \left(\frac{d}{d\lambda}\,r(\lambda)\right)^2 - \left(\sinh(r(\lambda))^4 - \sinh(r(\lambda))^2\right)\left(\frac{d}{d\lambda}\,\phi(\lambda)\right)^2 + 2\sqrt{2}\,\sinh(r(\lambda))^2\left(\frac{d}{d\lambda}\,t(\lambda)\right)\left(\frac{d}{d\lambda}\,\phi(\lambda)\right)\right) \quad \textbf{(17)}$$

```
> omega := 1;
```

$$\omega := 1 \quad \textbf{(18)}$$

The process of finding the Euler-Lagrange equations through substitutions and differentiations has been discussed in Ref. 2 and references therein:

```
> L1 := subs({t(lambda) = var1, diff(t(lambda), lambda) = var2, r(lambda) = var3, diff(r(lambda), lambda) = var4, phi(lambda) = var5, diff(phi(lambda), lambda) = var6}, L);
```

$$L1 := -var2^2 + var4^2 - \left(\sinh(var3)^4 - \sinh(var3)^2\right) var6^2 + 2\sqrt{2}\,\sinh(var3)^2\, var2\, var6 \quad \textbf{(19)}$$

```
> Epr11 := diff(L1, var2);
```

$$Epr11 := -2\, var2 + 2\sqrt{2}\,\sinh(var3)^2\, var6 \quad \textbf{(20)}$$

```
> Epr12 := diff(L1, var1);
```

$$Epr12 := 0 \quad \textbf{(21)}$$

```
> Epr13 := subs({var1 = t(lambda), var2 = diff(t(lambda), lambda), var3 = r(lambda), var4 = diff(r(lambda), lambda), var5 = phi(lambda), var6 = diff(phi(lambda), lambda)}, Epr11);
```

$$Epr13 := -2\,\frac{d}{d\lambda}\,t(\lambda) + 2\sqrt{2}\,\sinh(r(\lambda))^2\left(\frac{d}{d\lambda}\,\phi(\lambda)\right) \quad \textbf{(22)}$$

```
> Epr14 := subs({var1 = t(lambda), var2 = diff(t(lambda), lambda), var3 = r(lambda), var4 = diff(r(lambda), lambda), var5 = phi(lambda), var6 = diff(phi(lambda), lambda)}, Epr12);
```

$$Epr14 := 0 \quad \textbf{(23)}$$

```
> Epr15 := diff(Epr13, lambda);
```

$$Epr15 := -2\,\frac{d^2}{d\lambda^2}\,t(\lambda) + 4\sqrt{2}\,\sinh(r(\lambda))\left(\frac{d}{d\lambda}\,\phi(\lambda)\right)\left(\frac{d}{d\lambda}\,r(\lambda)\right)\cosh(r(\lambda)) + 2\sqrt{2}\,\sinh(r(\lambda))^2\left(\frac{d^2}{d\lambda^2}\,\phi(\lambda)\right) \quad \textbf{(24)}$$

```
> Eq16 := Epr15 - Epr14 = 0;
```

$$Eq16 := -2\,\frac{d^2}{d\lambda^2}\,t(\lambda) + 4\sqrt{2}\,\sinh(r(\lambda))\left(\frac{d}{d\lambda}\,\phi(\lambda)\right)\left(\frac{d}{d\lambda}\,r(\lambda)\right)\cosh(r(\lambda)) + 2\sqrt{2}\,\sinh(r(\lambda))^2\left(\frac{d^2}{d\lambda^2}\,\phi(\lambda)\right) = 0 \quad \textbf{(25)}$$

```
> Epr21 := diff(L1, var4);
```

$$Epr21 := 2\, var4 \tag{26}$$

```
> Epr22 := diff(L1, var3);
```

$$Epr22 := -\left(4 \sinh(var3)^3 \cosh(var3) - 2 \sinh(var3) \cosh(var3)\right) var6^2 + 4\sqrt{2} \sinh(var3)\, var2\, var6 \cosh(var3) \tag{27}$$

```
> Epr23 := subs({var1 = t(lambda), var2 = diff(t(lambda), lambda),
  var3 = r(lambda), var4 = diff(r(lambda), lambda), var5 = phi
  (lambda), var6 = diff(phi(lambda), lambda)}, Epr21);
```

$$Epr23 := 2 \frac{d}{d\lambda} r(\lambda) \tag{28}$$

```
> Epr24 := subs({var1 = t(lambda), var2 = diff(t(lambda), lambda),
  var3 = r(lambda), var4 = diff(r(lambda), lambda), var5 = phi
  (lambda), var6 = diff(phi(lambda), lambda)}, Epr22);
```

$$Epr24 := -\left(4 \sinh(r(\lambda))^3 \cosh(r(\lambda)) - 2 \sinh(r(\lambda)) \cosh(r(\lambda))\right) \left(\frac{d}{d\lambda} \phi(\lambda)\right)^2 + 4\sqrt{2} \sinh(r(\lambda)) \left(\frac{d}{d\lambda} t(\lambda)\right) \left(\frac{d}{d\lambda} \phi(\lambda)\right) \cosh(r(\lambda)) \tag{29}$$

```
> Epr25 := diff(Epr23, lambda);
```

$$Epr25 := 2 \frac{d^2}{d\lambda^2} r(\lambda) \tag{30}$$

```
> Eq26 := Epr25 - Epr24 = 0;
```

$$Eq26 := 2 \frac{d^2}{d\lambda^2} r(\lambda) + \left(4 \sinh(r(\lambda))^3 \cosh(r(\lambda)) - 2 \sinh(r(\lambda)) \cosh(r(\lambda))\right) \left(\frac{d}{d\lambda} \phi(\lambda)\right)^2 - 4\sqrt{2} \sinh(r(\lambda)) \left(\frac{d}{d\lambda} t(\lambda)\right) \left(\frac{d}{d\lambda} \phi(\lambda)\right) \cosh(r(\lambda)) = 0 \tag{31}$$

```
> Epr31 := diff(L1, var6);
```

$$Epr31 := -2 \left(\sinh(var3)^4 - \sinh(var3)^2\right) var6 + 2\sqrt{2} \sinh(var3)^2\, var2 \tag{32}$$

```
> Epr32 := diff(L1, var5);
```

$$Epr32 := 0 \tag{33}$$

```
> Epr33 := subs({var1 = t(lambda), var2 = diff(t(lambda), lambda),
  var3 = r(lambda), var4 = diff(r(lambda), lambda), var5 = phi
  (lambda), var6 = diff(phi(lambda), lambda)}, Epr31);
```

$$Epr33 := -2 \left(\sinh(r(\lambda))^4 - \sinh(r(\lambda))^2\right) \left(\frac{d}{d\lambda} \phi(\lambda)\right) + 2\sqrt{2} \sinh(r(\lambda))^2 \left(\frac{d}{d\lambda} t(\lambda)\right) \tag{34}$$

```
> Epr34 := subs({var1 = t(lambda), var2 = diff(t(lambda), lambda),
  var3 = r(lambda), var4 = diff(r(lambda), lambda), var5 = phi
  (lambda), var6 = diff(phi(lambda), lambda)}, Epr32);
```

$$Epr34 := 0 \tag{35}$$

```
> Epr35 := diff(Epr33, lambda);
```

$$Epr35 := -2 \left(4 \sinh(r(\lambda))^3 \left(\frac{d}{d\lambda} r(\lambda)\right) \cosh(r(\lambda)) - 2 \sinh(r(\lambda)) \left(\frac{d}{d\lambda} r(\lambda)\right) \cosh(r(\lambda))\right) \left(\frac{d}{d\lambda} \phi(\lambda)\right) - 2 \left(\sinh(r(\lambda))^4 - \sinh(r(\lambda))^2\right) \left(\frac{d^2}{d\lambda^2} \phi(\lambda)\right) \tag{36}$$

$$+4\sqrt{2}\sinh(r(\lambda))\left(\frac{d}{d\lambda}t(\lambda)\right)\left(\frac{d}{d\lambda}r(\lambda)\right)\cosh(r(\lambda))+2\sqrt{2}\sinh(r(\lambda))^2\left(\frac{d^2}{d\lambda^2}t(\lambda)\right)$$

```
> Eq36 := Epr35 - Epr34 = 0;
```

$$Eq36 := -2\left(4\sinh(r(\lambda))^3\left(\frac{d}{d\lambda}r(\lambda)\right)\cosh(r(\lambda))-2\sinh(r(\lambda))\left(\frac{d}{d\lambda}r(\lambda)\right)\cosh(r(\lambda))\right)\left(\frac{d}{d\lambda}\phi(\lambda)\right)-2\left(\sinh(r(\lambda))^4-\sinh(r(\lambda))^2\right)\left(\frac{d^2}{d\lambda^2}\phi(\lambda)\right)+4\sqrt{2}\sinh(r(\lambda))\left(\frac{d}{d\lambda}t(\lambda)\right)\left(\frac{d}{d\lambda}r(\lambda)\right)\cosh(r(\lambda))+2\sqrt{2}\sinh(r(\lambda))^2\left(\frac{d^2}{d\lambda^2}t(\lambda)\right)=0 \quad (37)$$

Set the initial conditions for massless particles:

```
> t0 := 0; r0 := 1/100; phi0 := Pi/10; dr0 := 1; dphi0 := 0; dt0 := dr0;
```

$$t0 := 0$$
$$r0 := \frac{1}{100}$$
$$\phi0 := \frac{\pi}{10}$$
$$dr0 := 1$$
$$dphi0 := 0$$
$$dt0 := 1 \quad (38)$$

Numerically solve the differential equations:

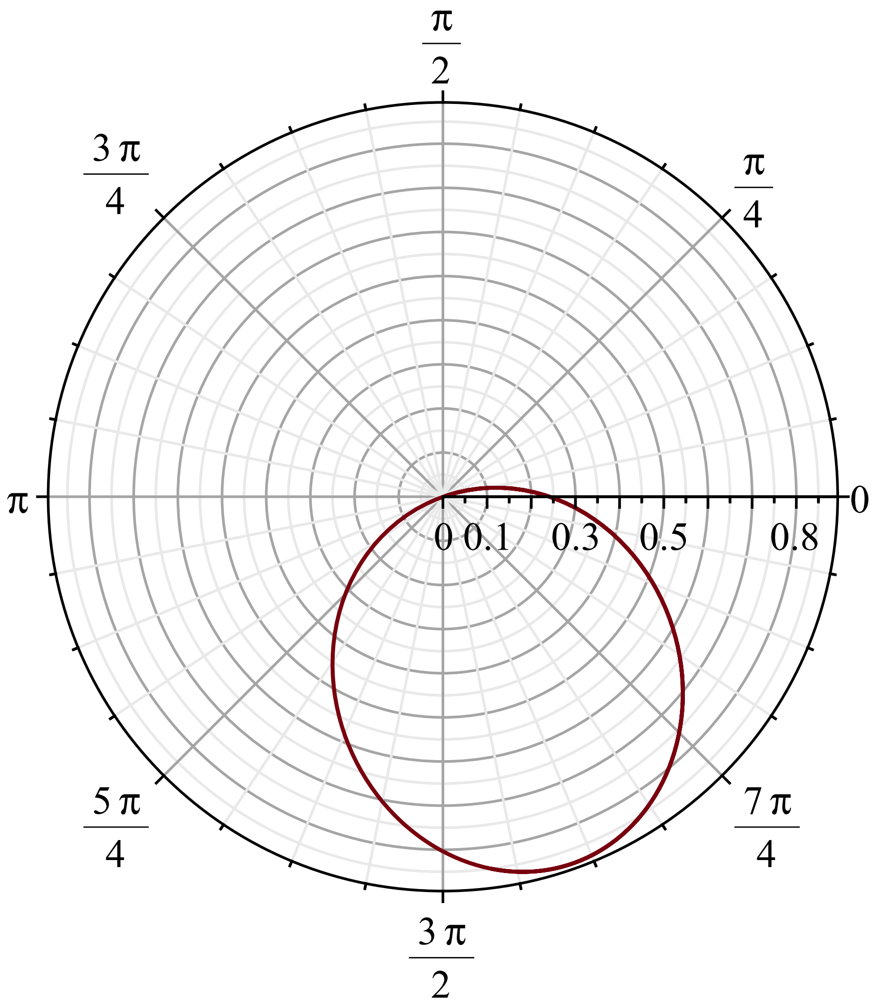
```
> soln0 := dsolve({Eq16, Eq26, Eq36, t(0) = t0, D(t)(0) = dt0, r(0) = r0, D(r)(0) = dr0, phi(0) = phi0, D(phi)(0) = dphi0}, {t(lambda), r(lambda), phi(lambda)}, numeric, output = operator);
```

soln0 := [λ = **proc**(lambda) ... **end proc**, φ = **proc**(lambda) ... **end proc**, D(φ) = **proc**(lambda)
...
end proc, *r* = **proc**(lambda) ... **end proc**, D(*r*) = **proc**(lambda) ... **end proc**, *t* = **proc**(lambda)
...
end proc, D(*t*) = **proc**(lambda) ... **end proc**] (39)

```
> with(plots):
```

Plot the trajectory of light in the xy-plane:

```
> polarplot([rhs(soln0(lambda)[4]), rhs(soln0(lambda)[2]), lambda = 0..5]);
```



When the phi-phi component of the metric is zero, calculated below, the light reaches the maximal radial distance:

```
> solve(sinh(r)^4 - sinh(r)^2 = 0, r);
```

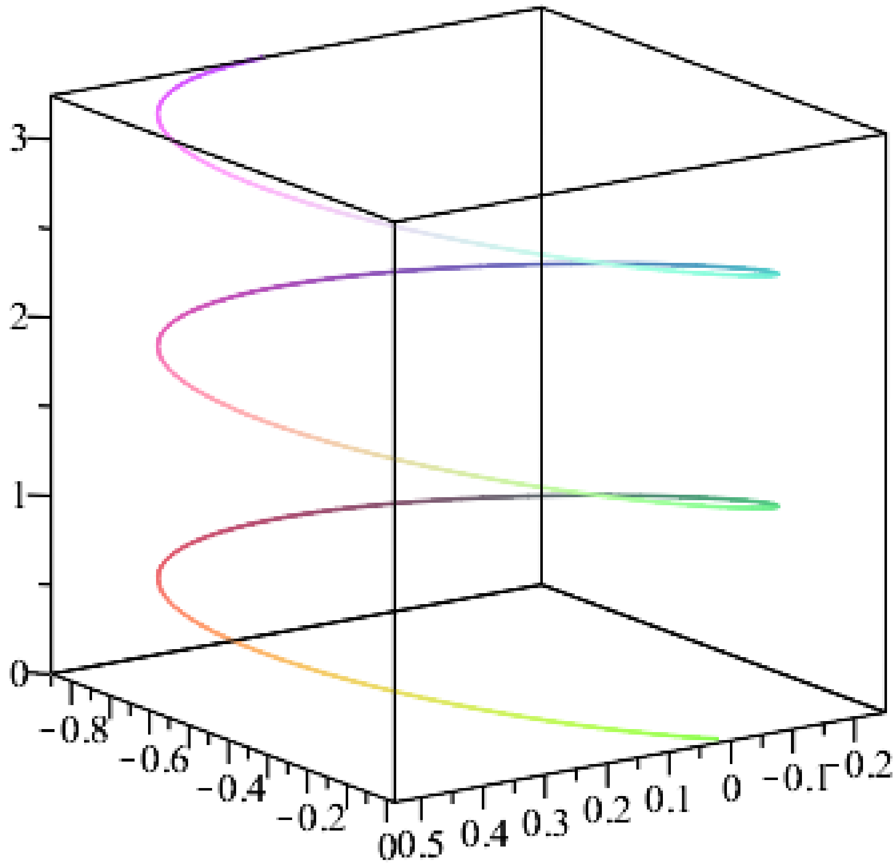
$$\ln\left(1+\sqrt{2}\right),\ -\ln\left(1+\sqrt{2}\right),\ 0 \quad \textbf{(40)}$$

```
> evalf(%);
```

$$0.8813735869,\ -0.8813735869,\ 0. \quad \textbf{(41)}$$

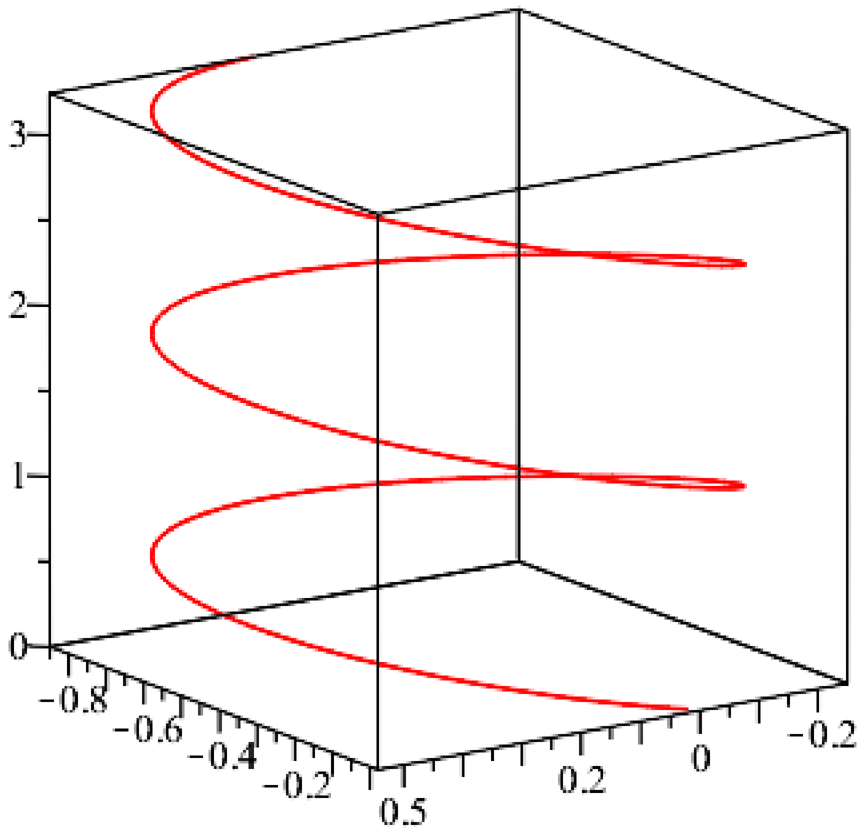
Plot and animate the trajectory of light in spacetime:

```
> spacecurve([rhs(soln0(lambda)[4])*cos(rhs(soln0(lambda)[2])), rhs
  (soln0(lambda)[4])*sin(rhs(soln0(lambda)[2])), rhs(soln0(lambda)
  [6])], lambda = 0..8);
```



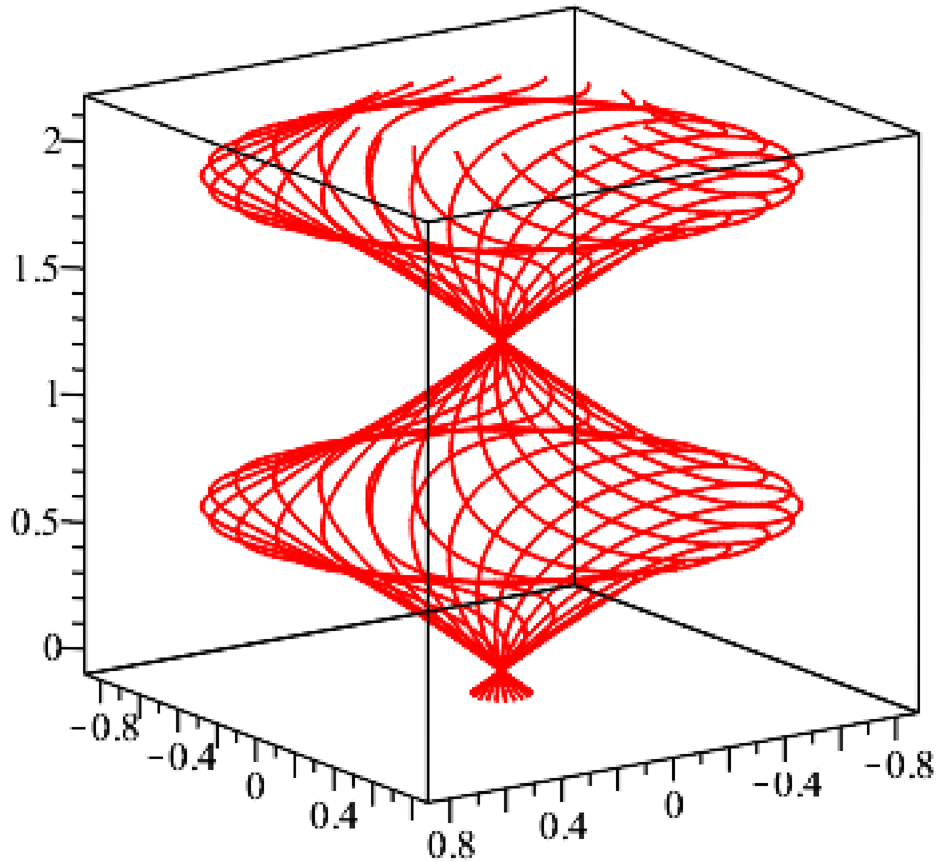

```
> animate(spacecurve,[[rhs(soln0(lambda)[4])*cos(rhs(soln0(lambda)
  [2])), rhs(soln0(lambda)[4])*sin(rhs(soln0(lambda)[2])), rhs(soln0
  (lambda)[6])], lambda = 0..s, color = red], s=0..8);
```

$s = 8.0000$



Produce a bunch of light trajectories:

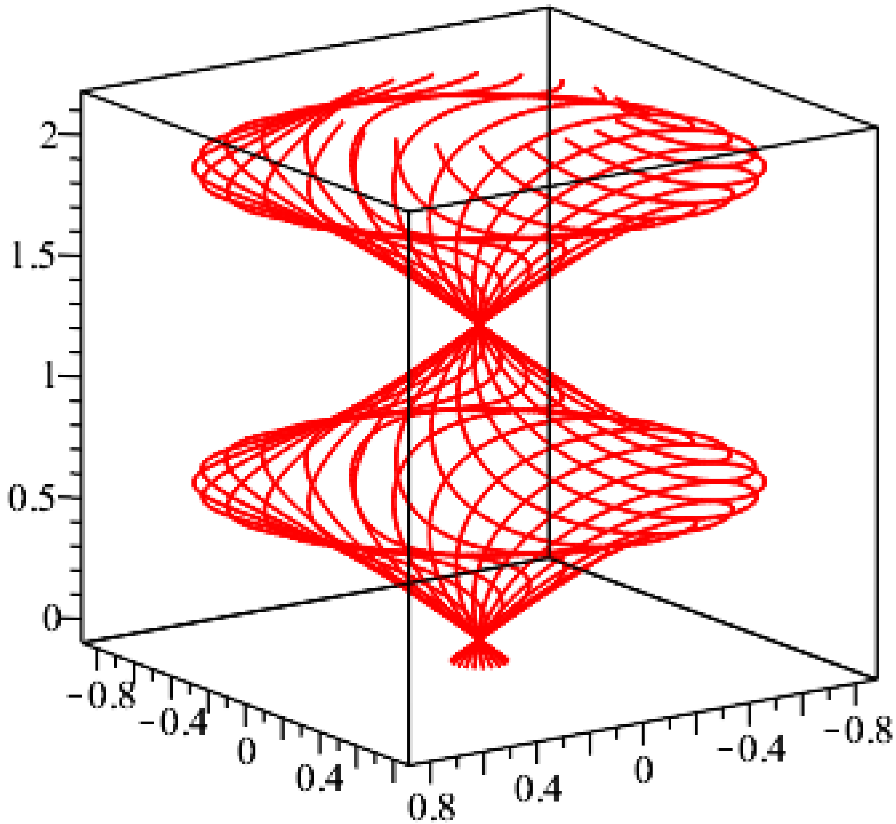
```
> for i from 1 to 20 do soln[i] := dsolve({Eq16, Eq26, Eq36, t(0) =
  t0, D(t)(0) = dt0, r(0) = r0, D(r)(0) = dr0, phi(0) = Pi/10*i, D
  (phi)(0) = dphi0}, {t(lambda), r(lambda), phi(lambda)}, numeric,
  output = operator); end do:
> for i from 1 to 20 do p[i] := spacecurve([rhs(soln[i](lambda)[4])*
  cos(rhs(soln[i](lambda)[2])), rhs(soln[i](lambda)[4])*sin(rhs(soln
  [i](lambda)[2])), rhs(soln[i](lambda)[6])], lambda = -1/10..5.8,
  color = red); end do:
> display(seq(p[i], i=1..20));
```



Create an animation:

```
> for i from 1 to 20 do ani[i] := animate(spacecurve, [[rhs(soln[i]
  (lambda)[4])*cos(rhs(soln[i](lambda)[2])), rhs(soln[i](lambda)[4])
  *sin(rhs(soln[i](lambda)[2])), rhs(soln[i](lambda)[6])], lambda =
  -1/10..s, color = red], s=-1/10..5.8); end do: #It will take a
  while
> display(seq(ani[i], i=1..20));
```

$s = 5.8000$



The null geodesics from a point on the axis diverge from the axis initially, reach a caustic at r = ln(1+ sqrt(2)), and then reconverge to a point.

References

1. S. W. Hawking and G. F. R. Ellis, *The large scale structure of space-time*, Cambridge University Press, Cambridge, 1973, Pp 168 - 170.
2. F. Wang, "Relativistic orbits and black holes," Maple Application Center, 2014. https://www.maplesoft.com/applications/view.aspx?SID=4508
2. F. Wang, *Physics with Maple*, Wiley, New York, 2006.