# Burgers' Vortex

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The Burgers vortex is a well known solution of the Navier Stokes equations that combines vorticity and shear. It allows for the study of a realistic flow in analytical form, thereby offering intuition for more complex flows. However, the cylindrical coordinate system makes certain calculations cumbersome to carry out by hand. This worksheet allows the user to explore aspects of the flow without having to carry out the calculations.

| Comments | Calculations |
|---|---|
| These are the three componenets of the Burgers vortex. | $ur := -\frac{1}{2}\cdot \text{alpha}\cdot r$ <br> $-\frac{1}{2}\alpha r$ **(1)** <br> $utheta := \frac{G0}{2\cdot \text{Pi}\cdot r}\cdot\left(1-\exp\left(-\frac{\text{alpha}\cdot r^2}{4\cdot \text{nu}}\right)\right)$ <br> $\frac{1}{2}\frac{G0\left(1-e^{-\frac{1}{4}\frac{\alpha r^2}{\nu}}\right)}{\pi r}$ **(2)** <br> $uz := \text{alpha}\cdot z$ <br> $\alpha z$ **(3)** |
| | *with*(*LinearAlgebra*) : <br> *with*(*plots*) : <br> *with*(*DEtools*) : <br> *with*(*Student*[*VectorCalculus*]) : <br><br> $v1 := VectorField(\langle ur, utheta, uz\rangle, cylindrical)$ |

$$-\frac{1}{2}\,\alpha\, r\bar{e}_r + \frac{1}{2}\,\frac{G0\left(1 - e^{-\frac{1}{4}\,\frac{\alpha r^2}{\nu}}\right)}{\pi r}\bar{e}_\theta + (\alpha z)\bar{e}_z \qquad \textbf{(4)}$$

$v1ev := subs(G0 = 10, \text{alpha} = 1, \text{nu} = 1e-6, v1)$

$$-\frac{1}{2}\, r\bar{e}_r + \frac{5\left(1 - e^{-2.500000000\;10^5\, r^2}\right)}{\pi r}\bar{e}_\theta + (z)\bar{e}_z \qquad \textbf{(5)}$$

$fp := fieldplot3d(v1ev, r = -10\,..10, \text{theta} = 0\,..\text{Pi}, z = -14\,..12, coords = cylindrical, thickness = 2, labels = ['r', '\text{theta}', 'z']) :$

$$sol3 := dsolve\left(\left\{diff(r(t), t) = -\frac{1}{2}\cdot 1\cdot r(t), diff(\text{theta}(t), t) = \frac{10}{2\cdot \text{Pi}\cdot r(t)}\cdot\left(1 - \exp\left(-\frac{1\cdot r(t)^2}{4\cdot 1e-6}\right)\right), diff(z(t), t) = 1\cdot z(t), r(0) = -2, \text{theta}(0) = 2\cdot\text{Pi}, z(0) = 0.08\right\}, [r(t), \text{theta}(t), z(t)], type = numeric, method = rkf45, output = array\left(\left[seq\left(\frac{k\cdot 5.1}{100}, k = 0\,..100\right)\right]\right)\right) :$$

$p2 := pointplot3d(Column(sol3(1)[2, 1], 2), Column(sol3(1)[2, 1], 3), Column(sol3(1)[2, 1], 4), coords = cylindrical, connect, colour = black, thickness = 3) :$

$$sol5 := dsolve\left(\left\{diff(r(t), t) = -\frac{1}{2}\cdot 1\cdot r(t), diff(\text{theta}(t), t) = \frac{10}{2\cdot \text{Pi}\cdot r(t)}\cdot\left(1 - \exp\left(-\frac{1\cdot r(t)^2}{4\cdot 1e-6}\right)\right), diff(z(t), t) = 1\cdot z(t), r(0) = -10, \text{theta}(0) = 0, z(0) = 1\right\}, [r(t), \text{theta}(t), z(t)], type = numeric, method = rkf45, output = array\left(\left[seq\left(\frac{k\cdot 2.5}{100}, k = 0\,..100\right)\right]\right)\right) :$$

$p4 := pointplot3d(Column(sol5(1)[2, 1], 2), Column(sol5(1)[2, 1], 3), Column(sol5(1)[2, 1], 4), coords = cylindrical, connect, colour = black, thickness = 3) :$

$$sol6 := dsolve\left(\left\{diff(r(t), t) = -\frac{1}{2}\cdot 1\cdot r(t), diff(\text{theta}(t), t)\right.\right.$$

$$= \frac{10}{2 \cdot \mathrm{Pi} \cdot r(t)} \cdot \left(1 - \exp\left(-\frac{1 \cdot r(t)^2}{4 \cdot 1\mathrm{e}{-6}}\right)\right), \mathit{diff}(z(t), t) = 1 \cdot z(t), r(0) = 8, \mathrm{theta}(0) = 0, z(0) = -4\Big\}, [r(t), \mathrm{theta}(t), z(t)], \mathit{type} = \mathit{numeric}, \mathit{method} = \mathit{rkf45}, \mathit{output} = \mathit{array}\left(\left[\mathit{seq}\left(\frac{k \cdot 1.4}{100}, k = 0..100\right)\right]\right)\right) :$$

$$p5 := \mathit{pointplot3d}(\mathit{Column}(\mathit{sol6}(1)[2, 1], 2), \mathit{Column}(\mathit{sol6}(1)[2, 1], 3), \mathit{Column}(\mathit{sol6}(1)[2, 1], 4), \mathit{coords} = \mathit{cylindrical}, \mathit{connect}, \mathit{colour} = \mathit{black}, \mathit{thickness} = 3) :$$

$$\mathit{sol7} := \mathit{dsolve}\left(\left\{\mathit{diff}(r(t), t) = -\frac{1}{2} \cdot 1 \cdot r(t), \mathit{diff}(\mathrm{theta}(t), t) = \frac{10}{2 \cdot \mathrm{Pi} \cdot r(t)} \cdot \left(1 - \exp\left(-\frac{1 \cdot r(t)^2}{4 \cdot 1\mathrm{e}{-6}}\right)\right), \mathit{diff}(z(t), t) = 1 \cdot z(t), r(0) = 3, \mathrm{theta}(0) = -2 \cdot \mathrm{Pi}, z(0) = -0.05\right\}, [r(t), \mathrm{theta}(t), z(t)], \mathit{type} = \mathit{numeric}, \mathit{method} = \mathit{rkf45}, \mathit{output} = \mathit{array}\left(\left[\mathit{seq}\left(\frac{k \cdot 5.8}{100}, k = 0..100\right)\right]\right)\right) :$$

$$p6 := \mathit{pointplot3d}(\mathit{Column}(\mathit{sol7}(1)[2, 1], 2), \mathit{Column}(\mathit{sol7}(1)[2, 1], 3), \mathit{Column}(\mathit{sol7}(1)[2, 1], 4), \mathit{coords} = \mathit{cylindrical}, \mathit{connect}, \mathit{colour} = \mathit{black}, \mathit{thickness} = 3) :$$

$$\mathit{display}([p2, p4, p5, p6, \mathit{fp}])$$

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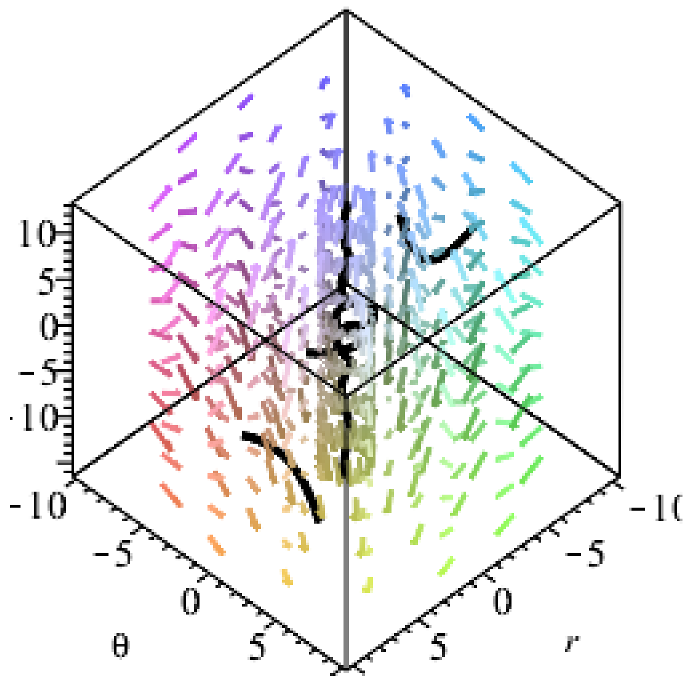
| | <br> |
| This is the divergence as a sanity check. | $mydiv := \frac{1}{r} \cdot diff(r \cdot ur, r) + \frac{1}{r} \cdot diff(utheta, \text{theta}) + diff(uz, z)$<br>$0$ **(6)** |
| Here is the vorticity. | $myvortr := \frac{1}{r} \cdot diff(uz, \text{theta}) - diff(utheta, z)$<br>$0$ **(7)**<br>$myvorttheta := diff(ur, z) - diff(uz, r)$<br>$0$ **(8)**<br>$myvortz := \frac{1}{r} \cdot diff(r \cdot utheta, r) - \frac{1}{r} \cdot diff(ur, \text{theta})$<br>$\frac{1}{4} \frac{G0\, \alpha\, e^{-\frac{1}{4} \frac{\alpha r^2}{\nu}}}{\nu \pi}$ **(9)** |
| These are the components of the rate of strain tensor. | $err := diff(ur, r)$<br>$-\frac{1}{2} \alpha$ **(10)**<br>$ethetatheta := \frac{1}{r} \cdot diff(utheta, \text{theta}) + \frac{ur}{r}$<br>$-\frac{1}{2} \alpha$ **(11)**<br>$ezz := diff(uz, z)$<br>**(12)** |

| | |
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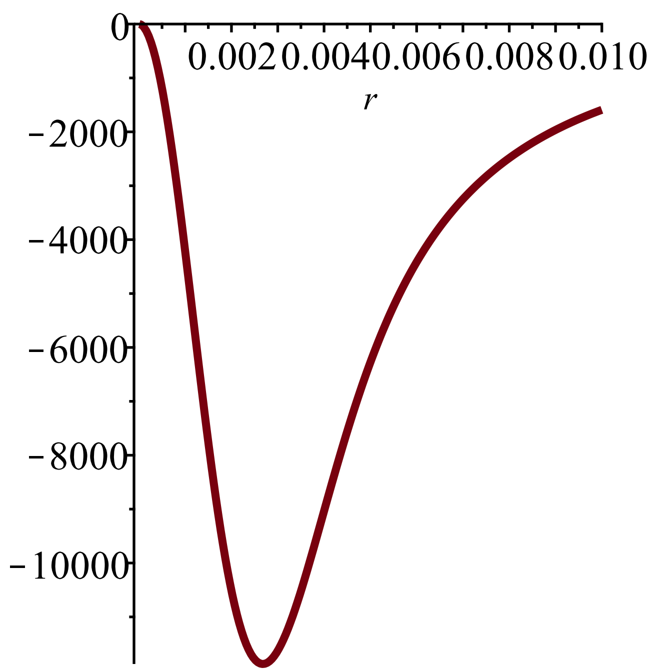
| | $\alpha$ **(12)** |
| | $ertheta := \frac{r}{2} \cdot diff\left(\frac{utheta}{r}, r\right) + \frac{1}{2 \cdot r} \cdot diff(ur, \text{theta})$ |
| | $\frac{1}{2} r \left( \frac{1}{4} \frac{G0\, \alpha\, e^{-\frac{1}{4} \frac{\alpha r^2}{\nu}}}{r \nu \pi} - \frac{G0 \left(1 - e^{-\frac{1}{4} \frac{\alpha r^2}{\nu}}\right)}{\pi r^3} \right)$ **(13)** |
| | $ethetaz := \frac{1}{2 \cdot r} \cdot diff(uz, \text{theta}) + \frac{1}{2} \cdot diff(utheta, z)$ |
| | $0$ **(14)** |
| | $erz := \frac{1}{2} \cdot diff(ur, z) + \frac{1}{2} \cdot diff(uz, r)$ |
| | $0$ **(15)** |
| This is the only non-zero shear component for a sample parameter set. | $myshearex := subs(G0 = 1, \text{alpha} = 1, \text{nu} = 1\text{e}{-6}, ertheta)$ |
| | $\frac{1}{2} r \left( \frac{79577.47152\, e^{-2.500000000\, 10^5 r^2}}{r} - \frac{1 - e^{-2.500000000\, 10^5 r^2}}{\pi r^3} \right)$ **(16)** |
| | $plot(myshearex, r = 0.00001 \,..\, 0.01, thickness = 3)$ |
| |  |
| | $with(linalg):$ |
| Now I define the actual rate of strain as | $e := Matrix(3, 3, [err, ertheta, erz, ertheta, ethetatheta, ethetaz, erz, ethetaz, ezz]):$ |

a matrix then use the Maple linear algebra package to define the second invariant (I used the wiki page for the scaling)

*trace*(*e*)

$$0 \tag{17}$$

$$myinvariant2 := \frac{1}{2} \cdot \left( trace(e)^2 - trace(Multiply(e, e)) \right)$$

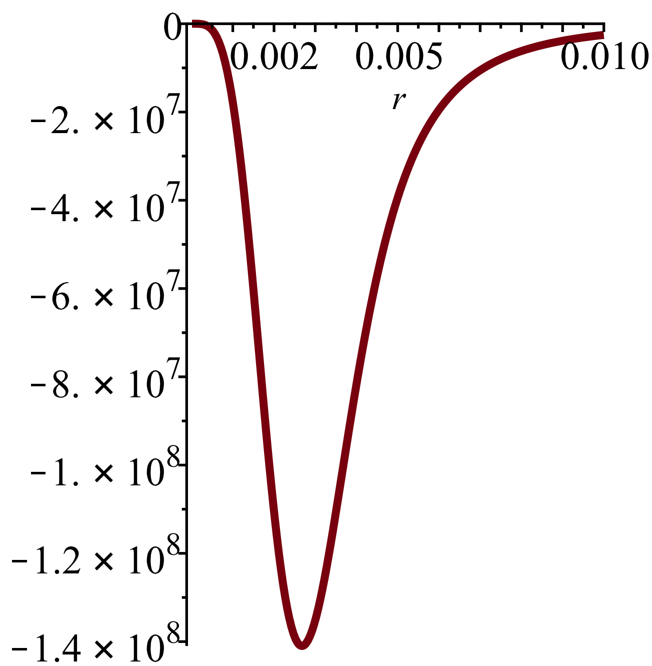
$$-\frac{3}{4}\alpha^2 - \frac{1}{4} r^2 \left( \frac{1}{4} \frac{G0\,\alpha\,e^{-\frac{1}{4}\frac{\alpha r^2}{\nu}}}{r\,\nu\,\pi} - \frac{G0\left(1 - e^{-\frac{1}{4}\frac{\alpha r^2}{\nu}}\right)}{\pi r^3} \right)^2 \tag{18}$$

$$myinvariant2ex := subs(G0 = 1, \mathrm{nu} = 1e{-}6, \mathrm{alpha} = 1, myinvariant2)$$

$$-\frac{3}{4} - \frac{1}{4} r^2 \left( \frac{79577.47152\, e^{-2.500000000\, 10^5 r^2}}{r} - \frac{1 - e^{-2.500000000\, 10^5 r^2}}{\pi r^3} \right)^2 \tag{19}$$

*plot*(*myinvariant2ex*, *r* = 0.00001 ..0.01, *thickness* = 3)

And here is a plot of the example showing that the second invariant has a local extremum for r>0




Now I define the enstrophy and plot a sample to show that in fact it is monotonically decreasing.

$$myens := myvortr^2 + myvorttheta^2 + myvortz^2$$

$$\frac{1}{16} \frac{G0^2\,\alpha^2 \left( e^{-\frac{1}{4}\frac{\alpha r^2}{\nu}} \right)^2}{\nu^2\,\pi^2} \tag{20}$$

*plot*(*subs*(*G0* = 1, nu = 1e−6, alpha = 1, *myens*), *r* = 0.00001

..0.01, *thickness* = 3 )