Eigenpairs

What are they and how are they found

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Fixed Points of a Function

A *fixed point* for the function $f(x)$ is a solution of the equation $f(x) = x$.

Find any fixed points for the function $f(x) = x^2 - 3x - 12$.

- Figure 1 contains a graph of f and the line $y = x$.
- The two intersections are the fixed points of f.
- Analytically, the fixed points are found by solving

$$x^2 - 3x - 12 = x$$

or

$$x^2 - 4x - 12 = 0$$

- The left-hand side factors to $(x + 2)(x - 6)$ so

$$x = -2 \text{ and } x = 6$$

are the fixed points of f.

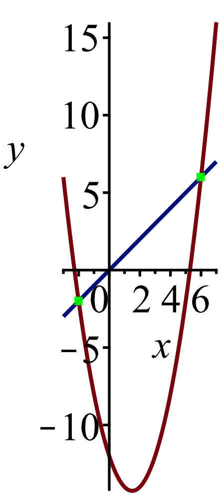


Figure 1 Graph of f and the line $y = x$

What Can a Matrix Leave Fixed?

$A = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$	$A\mathbf{x} = \mathbf{x}$ $\begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

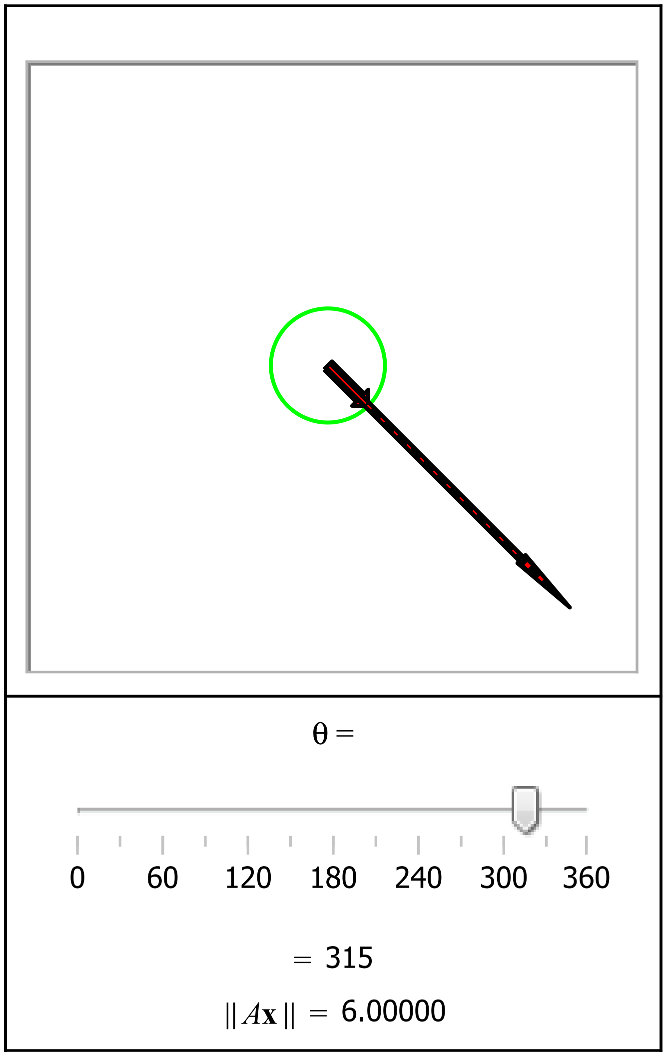
$A = \begin{bmatrix} 4 & 3 \\ -2 & -3 \end{bmatrix}$	$\|A\mathbf{x}\| = \|\mathbf{x}\|$ $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$; $\mathbf{v}_2 = \begin{bmatrix} 17 \\ -19 \end{bmatrix}$ $\|\mathbf{v}_1\| = \sqrt{2} = \left\| \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\| = \|A\mathbf{v}_1\|$; $\|\mathbf{v}_2\| = 5\sqrt{26} = \left\| \begin{bmatrix} 11 \\ 23 \end{bmatrix} \right\| = \|A\mathbf{v}_2\|$
$A = \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}$	$A\mathbf{x} = \lambda\,\mathbf{x}$ $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $\begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 6 \end{bmatrix} = 6 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Eigenpairs

What is an Eigenpair?

When a linear transformation represented by a matrix A maps a vector to a multiple of itself, it generates an invariant subspace: all vectors collinear with with that vector remain collinear with that vector, and consequently remain in that subspace. This notion of "fixed point" has proven to be the most useful. Vectors in such an invariant subspace are called *eigenvectors*, and the multiplicative factor by which such a vector is stretched (or shrunk), is called an *eigenvalue*. It is convenient to group an eigenvector and its corresponding eigenvalue as an *eigenpair*.

For the matrix $A = \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}$, the slider below the following figure controls the angle the unit vector inside the circle makes with the positive x-direction. As this vector is rotated, the vector $A\mathbf{x}$ (in black) correspondingly rotates. If the vectors $\mathbf{x}$ and $A\mathbf{x}$ are collinear, then $A\mathbf{x}$ is a multiple of $\mathbf{x}$ and $\mathbf{x}$ is in an invariant subspace of A.



Computing Eigenpairs

Initialization

Initialize	
• Tools>Load Package: Student Linear Algebra	Loading Student:-LinearAlgebra
• Control-drag $A=\ldots$ • Context Menu: Assign Name	$A = \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix} \xrightarrow{\text{assign}}$
• Context Menu: Assign Name	$\mathbf{x} = \langle a, b\rangle \xrightarrow{\text{assign}}$

Directly in Maple

Maple has built-in tools for obtaining eigenvalues and eigenvectors. The following example illustrates how to access these tools through the Context Menu.

Obtain the eigenpairs of A
• Type the name A. Context Menu: Evaluate and Display Inline • Context Menu: Student Linear Algebra≻Eigenvalues, etc.≻Eigenvectors • Context Menu: Select Element≻2 • Context Menu: Select Elements≻Split into Columns • Context Menu: Assign to a Name≻**X**
$A = \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix} \xrightarrow{\text{eigenvectors}} \begin{bmatrix} 6 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \xrightarrow{\text{select entry 2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \xrightarrow{\text{split into columns}}$ $\left[\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right] \xrightarrow{\text{assign to a name}} X$

The underlying commands in the *LinearAlgebra* packages would be **Eigenvalues** and **Eigenvectors**. These options exist separately in the Context Menu, but the choice "Eigenvectors" produces both eigenvalues and eigenvectors. The column vector returned contains the eigenvalues; the columns of the matrix to its right are the corresponding eigenvectors. The example shows how to access these columns by forming a list **X** of the eigenvectors, which can then be addressed as $\mathbf{X}_1$ and $\mathbf{X}_2$.

Note that successive calls to Eigenvectors option need not preserve the order in which the eigenpairs are presented.

Angle between **x** *and* A**x**

For what **x** is the angle between **x** and A**x** equal to zero?
• Form the sequence A**x** and **x**, then press the Enter key. • Context Menu: Student Linear Algebra≻Standard Operations≻Vector Angle • Context Menu: Conversions≻Equate to 0 • Context Menu: Solve≻Solve for Variable≻a

$A.\mathbf{x}, \mathbf{x}$ $$\begin{bmatrix} 5a - b \\ -a + 5b \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix}$$ $\xrightarrow{\text{angle between}}$ $$\arccos\left(\frac{(5a-b)a + (-a+5b)b}{\sqrt{(5a-b)^2 + (-a+5b)^2}\sqrt{a^2+b^2}}\right)$$ $\xrightarrow{\text{equate to 0}}$ $$\arccos\left(\frac{(5a-b)a + (-a+5b)b}{\sqrt{(5a-b)^2 + (-a+5b)^2}\sqrt{a^2+b^2}}\right) = 0$$ $\xrightarrow{\text{solve for a}}$ $$[[a = b], [a = -b]]$$
Determine x
• Write the name **x**. Context Menu: Evaluate and Display Inline • Context Menu: Evaluate at a Point>Set $a = b = 1$ (then set $b = 1$ and $a = -b = -1$) • Context Menu: Assign to a Name>v[1] (then v[2])

$\mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix} \xrightarrow{\text{evaluate at point}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\xrightarrow{\text{assign to a name}} \mathbf{v}_1$	$\mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix} \xrightarrow{\text{evaluate at point}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\xrightarrow{\text{assign to a name}} \mathbf{v}_2$

Deduce the eigenvalues
• Write the product of A with $\mathbf{v}_k$, $k = 1, 2$. Context Menu: Evaluate and Display Inline

$A.\mathbf{v}_1 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$	$A.\mathbf{v}_2 = \begin{bmatrix} -6 \\ 6 \end{bmatrix}$

Any vector whose direction is that of $\mathbf{v}_1$ is stretched by a factor of 4; the eigenvalue for this eigenvector is therefore 4.

Any vector whose direction is that of $\mathbf{v}_2$ is stretched by a factor of 6; the eigenvalue for this eigenvector is therefore 6.

Collinearity of **x** *and* A**x**

The vectors **x** and A**x** will be collinear if these vectors are proportional.

This is expressed analytically by the equality $A\mathbf{x} = \lambda\,\mathbf{x}$ for some scalar λ.

Any vector **x** for which such an equality holds is an *eigenvector*, and the corresponding scalar λ is an *eigenvalue*.

<table>
<tr><th colspan="2">Form the equations implied by $A\mathbf{x} = \lambda\,\mathbf{x}$</th></tr>
<tr><td>• Write the sequence of vectors; press the Enter key.

• Context Menu: Equate

• Context Menu: Assign to a Name>E</td><td>$A.\mathbf{x},\ \lambda\,\mathbf{x}$
$$\begin{bmatrix} 5a - b \\ -a + 5b \end{bmatrix}, \begin{bmatrix} \lambda a \\ \lambda b \end{bmatrix}$$
$\xrightarrow{\text{equate}}$
$$[5a - b = \lambda a,\ -a + 5b = \lambda b]$$
$\xrightarrow{\text{assign to a name}}$
$$E$$</td></tr>
<tr><th colspan="2">Graphically explore the λ-dependence of the lines $[5a - b = \lambda a,\ -a + 5b = \lambda b]$</th></tr>
<tr><td>• Click the button labeled "Graphic"</td><td>• Graphic</td></tr>
<tr><th colspan="2">Solve the equations emanating from the equality $A\mathbf{x} = \lambda\,\mathbf{x}$</th></tr>
<tr><td colspan="2">• Type the name E and press the Enter key.
• Context Menu: Solve>Eliminate a Variable>lambda
• Context Menu: Select Element>2
• Context Menu: Solve>Solve for Variables>a</td></tr>
<tr><td colspan="2">E
$$[5a - b = \lambda a,\ -a + 5b = \lambda b]$$
$\xrightarrow{\text{eliminate lambda}}$
$$\left[\left\{\lambda = -\frac{a - 5b}{b}\right\},\ \{-a^2 + b^2\}\right]$$
$\xrightarrow{\text{select entry 2}}$
$$\{-a^2 + b^2\}$$
$\xrightarrow{\text{solve (specified)}}$
$$\{a = b\},\ \{a = -b\}$$</td></tr>
</table>

The system $A\mathbf{x} = \lambda\,\mathbf{x}$ represents two equations in the two unknowns a and b, and the parameter λ. To what extent can a and b be obtained as functions of λ, that is, as $a(\lambda), b(\lambda)$? The following display summarizes the relationships between the variables and the parameter λ.

$$\begin{cases} \lambda = 4 & a = b & \mathbf{x} = a\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \lambda = 6 & a = -b & \mathbf{x} = a\begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ \text{else} & a = b = 0 & \mathbf{x} = \mathbf{0} \end{cases}$$

From this display, it should be inferred that the functional relationships between the variables and the parameter λ will not be given by simple expressions. The following Maple calculations show this.

Seeking $a = a(\lambda)$ and $b = b(\lambda)$	
• Reference the first equation and press the Enter key. • Context Menu: Solve>Isolate Expression for>b • Context Menu: Substitute Into>E[2] • Context Menu: Move to Left • Context Menu: Simplify>Simplify	
E_1 $5\,a - b = \lambda\,a$ $\xrightarrow{\text{isolate for b}}$ $b = -\lambda\,a + 5\,a$ $\xrightarrow{\text{substitute into}}$ $-5\,\lambda\,a + 24\,a = \lambda\,(-\lambda\,a + 5\,a)$ $\xrightarrow{\text{move to left}}$ $-5\,\lambda\,a + 24\,a - \lambda\,(-\lambda\,a + 5\,a) = 0$ $\overset{\text{simplify}}{=}$ $a\,(\lambda - 4)\,(\lambda - 6) = 0$	
• Control-drag the equation $b = -\lambda\,a + 5\,a$ • Context Menu: Evaluate at a Point>$\lambda = 4$ (then $\lambda = 6$)	
$b = -\lambda\,a + 5\,a \xrightarrow{\text{evaluate at point}} b = a$	$b = -\lambda\,a + 5\,a \xrightarrow{\text{evaluate at point}} b = -a$

Solution of $(A - \lambda I)\mathbf{x} = \mathbf{0}$

Rearrange the equation $A\mathbf{x} = \lambda\,\mathbf{x}$ to $A\mathbf{x} - \lambda\,\mathbf{x} = \mathbf{0}$, and then further to $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

$$A - \lambda I = \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 - \lambda & -1 \\ -1 & 5 - \lambda \end{bmatrix}$$

Hence, an eigenvector **x** is a member of the null space of the matrix $A - \lambda I$, where λ is a known eigenvalue.

Implement this rearrangement in Maple

- Enter the label for the system equations,
  Context Menu: Evaluate and Display Inline
- Context Menu: Student Linear Algebra>Constructions>Generate Matrix>Matrix-Vector pair

$$E = [5\,a - b = \lambda\,a,\ -a + 5\,b = \lambda\,b] \xrightarrow{\text{to Matrix form}} \begin{bmatrix} -\lambda + 5 & -1 \\ -1 & -\lambda + 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Obtain bases for the null spaces of $A - \lambda_k I$, $k = 1, 2$

- Write $A - 4\,I$ (which, in Maple, can be shortened to $A - 4$).
  Context Menu: Evaluate and Display Inline
- Context Menu: Student Linear Algebra>Vector Spaces>Null Space
- Repeat for $A - 6\,I$.

$$A - 4 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \xrightarrow{\text{null space}} \left[\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right] \qquad A - 6 = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \xrightarrow{\text{null space}} \left[\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$

Solution from First Principles

The equations emanating from the equality $(A - \lambda I)\mathbf{x} = \mathbf{0}$ are homogeneous, so **x** is the zero vector unless $|A - \lambda I|$, the determinant of $A - \lambda I$, is zero, in which case any nonzero solution **x** is an eigenvector. (Recall the Graphic .)

The polynomial $|A - \lambda I|$ is called the *characteristic polynomial*, and its zeros are the eigenvalues.

Obtain the *characteristic polynomial* and its zeros, the eigenvalues

- Write $A - \lambda I$, where I is the 2×2 identity matrix obtained via the Matrix palette.
  Context Menu: Evaluate and Display Inline
- Context Menu: Student Linear Algebra>Standard Operations>Determinant
- Context Menu: Solve>Solve

$$A - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 - \lambda & -1 \\ -1 & 5 - \lambda \end{bmatrix} \xrightarrow{\text{determinant}} \lambda^2 - 10\,\lambda + 24 \xrightarrow{\text{solve}} \{\lambda = 6\}, \{\lambda = 4\}$$

Alternatively, obtain the characteristic polynomial directly
• Write the name A. Context Menu: Evaluate and Display Inline • Context Menu: Student Linear Algebra>Eigenvalues, etc.>Characteristic Polynomial
$A = \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix} \xrightarrow{\text{characteristic polynomial}} \lambda^2 - 10\,\lambda + 24$
Extract the eigenvectors from the matrices $A - \lambda_k I$, $k = 1, 2$
• Write $A - \lambda_k$ for $\lambda_1 = 4$ and $\lambda_2 = 6$. Context Menu: Evaluate and Display Inline • Context Menu: Student Linear Algebra>Solvers and Forms>Row Echelon Form>Reduced

$A - 4 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \xrightarrow{\text{reduced row echelon form}} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$	$A - 6 = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \xrightarrow{\text{reduced row echelon form}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$
$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow a - b = 0$ $a - b = 0 \Rightarrow a = b \Rightarrow \mathbf{x} = \begin{bmatrix} a \\ a \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow a + b = 0$ $a + b = 0 \Rightarrow a = -b \Rightarrow \mathbf{x} = \begin{bmatrix} -b \\ b \end{bmatrix}$ $= b \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Three-Dimensional Example

Initialization

Initialize	
• Click the *restart* icon in the toolbar, or execute the **restart** command to the right.	*restart* :
• Tools>Load Package: Student Linear Algebra	Loading Student:-LinearAlgebra

• Context Menu: Assign Name	$A = \begin{bmatrix} -2 & 2 & 3 \\ -2 & 3 & 2 \\ -4 & 2 & 5 \end{bmatrix} \xrightarrow{\text{assign}}$
• Context Menu: Assign Name	$\mathbf{x} = \langle a, b, c \rangle \xrightarrow{\text{assign}}$

Eigenpairs Directly in Maple

• Write the name A. Context Menu: Evaluate and Display Inline • Context Menu: Student Linear Algebra≻Eigenvalues, etc.≻Eigenvectors • Context Menu: Select Element≻2 • Context Menu: Select Elements≻Split into Columns
$A = \begin{bmatrix} -2 & 2 & 3 \\ -2 & 3 & 2 \\ -4 & 2 & 5 \end{bmatrix} \xrightarrow{\text{eigenvectors}} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{\text{select entry 2}} \begin{bmatrix} \frac{1}{2} & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{\text{split into columns}} \left[\begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right]$

Collinearity via Vanishing of the Cross Product

Form and solve $A\mathbf{x} \times \mathbf{x} = \mathbf{0}$
• Common Symbols palette: Cross-product operator Press Enter key. • Context Menu: Conversions≻To List • Context Menu: Solve≻Solve • Context Menu: Assign to a Name≻S

$A.\mathbf{x} \times \mathbf{x}$

$$\begin{bmatrix} (-2a+3b+2c)c-(-4a+2b+5c)b \\ -(-2a+2b+3c)c+(-4a+2b+5c)a \\ (-2a+2b+3c)b-(-2a+3b+2c)a \end{bmatrix}$$

$\xrightarrow{\text{to list}}$

$[(-2a+3b+2c)c-(-4a+2b+5c)b, -(-2a+2b+3c)c+(-4a+2b+5c)a, (-2a+2b+3c)b-(-2a+3b+2c)a]$

$\xrightarrow{\text{solve}}$

$$\{a=a, b=0, c=a\}, \{a=a, b=2a, c=0\}, \{a=a, b=a, c=a\}$$

$\xrightarrow{\text{assign to a name}}$

$$S$$

Obtain the eigenvectors $\mathbf{v}_k$, $k=1, 2, 3$

- Write the name of one of the three solutions.
  Context Menu: Evaluate and Display Inline
- Context Menu: Substitute Into
- Context Menu: Evaluate at a Point > $a=1$
- Context Menu: Assign to a Name > v[k], $k=1, 2, 3$, as appropriate

$$S_1 = \{a=a, b=0, c=a\} \xrightarrow{\text{substitute into}} \begin{bmatrix} a \\ 0 \\ a \end{bmatrix} \xrightarrow{\text{evaluate at point}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \xrightarrow{\text{assign to a name}} \mathbf{v}_1$$

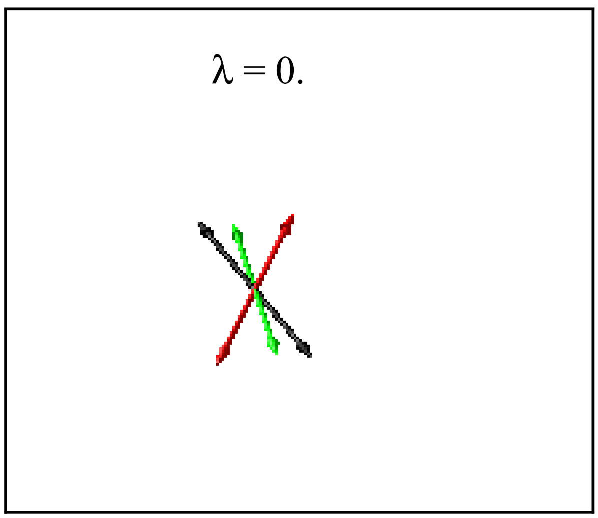
$$S_2 = \{a=a, b=2a, c=0\} \xrightarrow{\text{substitute into}} \begin{bmatrix} a \\ 2a \\ 0 \end{bmatrix} \xrightarrow{\text{evaluate at point}} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \xrightarrow{\text{assign to a name}} \mathbf{v}_2$$

$$S_3 = \{a=a, b=a, c=a\} \xrightarrow{\text{substitute into}} \begin{bmatrix} a \\ a \\ a \end{bmatrix} \xrightarrow{\text{evaluate at point}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \xrightarrow{\text{assign to a name}} \mathbf{v}_3$$

Infer the eigenvalues from $A\mathbf{v}_k = \lambda_k \mathbf{v}_k$

$A.\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$	$A.\mathbf{v}_2 = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$	$A.\mathbf{v}_3 = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$

Collinearity by Proportionality

Form the equations $A\mathbf{x} = \lambda\, \mathbf{x}$
Write a sequence of the two vectors, and press the Enter key. Context Menu: Equate Context Menu: Assign to a Name>E
$A.\mathbf{x},\ \lambda\, \mathbf{x}$ $$\begin{bmatrix} -2a+2b+3c \\ -2a+3b+2c \\ -4a+2b+5c \end{bmatrix}, \begin{bmatrix} \lambda a \\ \lambda b \\ \lambda c \end{bmatrix}$$ $\xrightarrow{\text{equate}}$ $$[-2a+2b+3c=\lambda a,\ -2a+3b+2c=\lambda b,\ -4a+2b+5c=\lambda c]$$ $\xrightarrow{\text{assign to a name}}$ $$E$$
Explore (graphically) the intersection of three planes through the origin

Solve the equations analytically
• Type E, the name of the list of equations, and press the Enter key. • Context Menu: Solve>Eliminate a Variable>lambda • Context Menu: Select Element>2 • Context Menu: Solve>Solve for Variables>b, c • Context Menu: Assign to a Name>Q

E

$$\left[-2\,a+2\,b+3\,c=\lambda\,a,\ -2\,a+3\,b+2\,c=\lambda\,b,\ -4\,a+2\,b+5\,c=\lambda\,c\right]$$

$\xrightarrow{\text{eliminate lambda}}$

$$\left[\left\{\lambda=-\frac{2\,a-3\,b-2\,c}{b}\right\},\ \left\{4\,a\,b-2\,a\,c-2\,b^2-2\,b\,c+2\,c^2,\ -2\,a^2+5\,a\,b+2\,a\,c-2\,b^2-3\,b\,c\right\}\right]$$

$\xrightarrow{\text{select entry 2}}$

$$\left\{4\,a\,b-2\,a\,c-2\,b^2-2\,b\,c+2\,c^2,\ -2\,a^2+5\,a\,b+2\,a\,c-2\,b^2-3\,b\,c\right\}$$

$\xrightarrow{\text{solve (specified)}}$

$$\{b=0,\ c=a\},\ \{b=2\,a,\ c=0\},\ \{b=a,\ c=a\}$$

$\xrightarrow{\text{assign to a name}}$

$$Q$$

Obtain the eigenvectors

- Write the name of one of the three solutions.
  Context Menu: Evaluate and Display Inline
- Context Menu: Substitute Into
- Context Menu: Evaluate at a Point>$a=1$

$$Q_1=\{b=0,\ c=a\}\xrightarrow{\text{substitute into}}\begin{bmatrix}a\\0\\a\end{bmatrix}\xrightarrow{\text{evaluate at point}}\begin{bmatrix}1\\0\\1\end{bmatrix}$$

$$Q_2=\{b=2\,a,\ c=0\}\xrightarrow{\text{substitute into}}\begin{bmatrix}a\\2\,a\\0\end{bmatrix}\xrightarrow{\text{evaluate at point}}\begin{bmatrix}1\\2\\0\end{bmatrix}$$

$$Q_3=\{b=a,\ c=a\}\xrightarrow{\text{substitute into}}\begin{bmatrix}a\\a\\a\end{bmatrix}\xrightarrow{\text{evaluate at point}}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

Eigenvectors as Members of the Null Space of $A-\lambda_k\,I$

- Write $A-1$ (which Mapleunderstands to be $A-1\,I$).
  Context Menu: Evaluate and Display Inline
- Context Menu: Student Linear Algebra>Vector Spaces>Null Space
- Repeat for $A-2\,I$, and $A-3\,I$.

$A - 1 = \begin{bmatrix} -3 & 2 & 3 \\ -2 & 2 & 2 \\ -4 & 2 & 4 \end{bmatrix} \xrightarrow{\text{null space}} \left[\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right]$	$A - 2 = \begin{bmatrix} -4 & 2 & 3 \\ -2 & 1 & 2 \\ -4 & 2 & 3 \end{bmatrix} \xrightarrow{\text{null space}} \left[\begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}\right]$	$A - 3 = \begin{bmatrix} -5 & 2 & 3 \\ -2 & 0 & 2 \\ -4 & 2 & 2 \end{bmatrix} \xrightarrow{\text{null space}} \left[\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$

Solution from First Principles

Obtain the *characteristic polynomial* and its zeros, the eigenvalues
• Write $A - \lambda I$, where I is the 3×3 identity matrix obtained via the Matrix palette. Context Menu: Evaluate and Display Inline • Context Menu: Student Linear Algebra≻Standard Operations≻Determinant • Context Menu: Solve≻Solve
$A - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2-\lambda & 2 & 3 \\ -2 & 3-\lambda & 2 \\ -4 & 2 & 5-\lambda \end{bmatrix} \xrightarrow{\text{determinant}} -\lambda^3 + 6\lambda^2 - 11\lambda + 6 \xrightarrow{\text{solve}}$ $\{\lambda = 1\}, \{\lambda = 2\}, \{\lambda = 3\}$
Alternatively, obtain the characteristic polynomial directly
• Write the name A. Context Menu: Evaluate and Display Inline • Context Menu: Student Linear Algebra≻Eigenvalues, etc.≻Characteristic Polynomial
$A = \begin{bmatrix} -2 & 2 & 3 \\ -2 & 3 & 2 \\ -4 & 2 & 5 \end{bmatrix} \xrightarrow{\text{characteristic polynomial}} \lambda^3 - 6\lambda^2 + 11\lambda - 6$
Extract the eigenvectors from the matrices $A - \lambda_k I$, $k = 1, 2, 3$
• Write $A - \lambda_k$ for $\lambda_k = k$, $k = 1, 2, 3$. Context Menu: Evaluate and Display Inline • Context Menu: Student Linear Algebra≻Solvers and Forms≻Row Echelon Form≻Reduced

$A - 1 = \begin{bmatrix} -3 & 2 & 3 \\ -2 & 2 & 2 \\ -4 & 2 & 4 \end{bmatrix}$ $\xrightarrow{\text{reduced row echelon form}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow a - c = 0, b = 0 \Rightarrow \mathbf{x} = \begin{bmatrix} a \\ 0 \\ a \end{bmatrix}$ $= a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$
$A - 2 = \begin{bmatrix} -4 & 2 & 3 \\ -2 & 1 & 2 \\ -4 & 2 & 3 \end{bmatrix}$ $\xrightarrow{\text{reduced row echelon form}} \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow a - b/2 = 0, c = 0 \Rightarrow$ $\mathbf{x} = \begin{bmatrix} a \\ 2\,a \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$
$A - 3 = \begin{bmatrix} -5 & 2 & 3 \\ -2 & 0 & 2 \\ -4 & 2 & 2 \end{bmatrix}$ $\xrightarrow{\text{reduced row echelon form}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow a - c = 0, b - c = 0 \Rightarrow \mathbf{x}$ $= \begin{bmatrix} c \\ c \\ c \end{bmatrix} = c \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Glimpses Beyond

Example 1: Repeated Eigenvalue

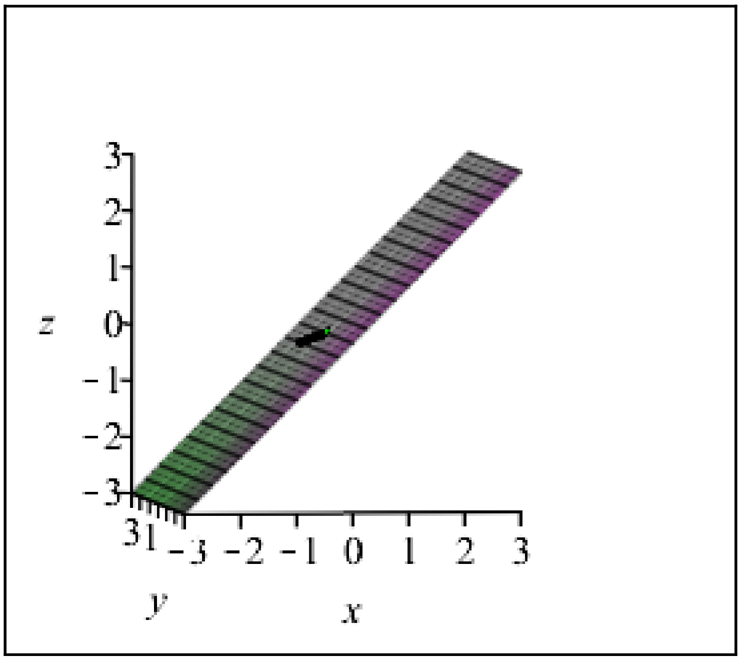
The matrix $A := \begin{bmatrix} 1 & 0 & 1 \\ -3 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$: has eigenpairs $1, \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ and $2, \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

In other words, there are two eigenvalues, 1 and 2.

The eigenvalue 2 has two eigenvectors associated with it, so every vector in the plane spanned by $\mathbf{i} + \mathbf{k}$ and $\mathbf{j}$ are eigenvectors with eigenvalue 2.

• Define a general linear combination of the two eigenvectors associated with the eigenvalue 2.	$\mathbf{Y} := a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \\ a \end{bmatrix}$
• Calculate $A\mathbf{Y}$ and compare to $2\,\mathbf{Y}$.	$A.\mathbf{Y} = \begin{bmatrix} 2\,a \\ 2\,b \\ 2\,a \end{bmatrix}$

In the following sketch, the plane is the invariant subspace determined by the span of the eigenvectors $\mathbf{i} + \mathbf{k}$ and $\mathbf{j}$; the line is the invariant subspace determined by the eigenvector $\mathbf{i} + 3\,\mathbf{j}$.



Example 2: A Deficient Case

The matrix $A := \begin{bmatrix} 5 & -5 & -4 \\ -4 & 8 & 5 \\ 7 & -11 & -7 \end{bmatrix}$: has the single eigenpair $2, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$.

The transition matrix $Q = \begin{bmatrix} -3 & -12 & -1 \\ 3 & 15 & 1 \\ -6 & -27 & 1 \end{bmatrix}$ will reduce A to its Jordan form $Q^{-1} A\, Q = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.

The first column of Q is a multiple of the single eigenvector. The second and third columns are *generalized eigenvectors* $\mathbf{b}_2$ and $\mathbf{b}_3$.

The vector $\mathbf{b}_3$ must be found first, and then $\mathbf{b}_2 = C\,\mathbf{b}_3$ and $\mathbf{b}_1 = C\,\mathbf{b}_2$, where $C = A - 2\,I$.

▶ *Theory*

▼ *Visualization*

- In Figure 1 , the black line is the eigenspace M_1.
- The two green vectors are a basis for M_2, itself shown as the yellow plane.
- The red vector is a vector in M_3 that is not in M_2.

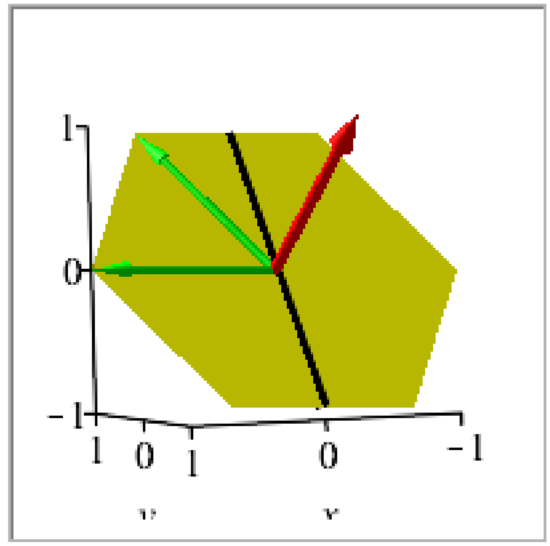


Figure 1 $M_1 \subset M_2 \subset M_3$

▼ *Choice of* $\mathbf{b}_3$ *and Its Consequence*

Taking the red vector as $\mathbf{b}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ gives $\mathbf{b}_2 = C\,\mathbf{b}_3 = \begin{bmatrix} -12 \\ 15 \\ -27 \end{bmatrix}$, and

$\mathbf{b}_1 = C\,\mathbf{b}_2 = \begin{bmatrix} -3 \\ 3 \\ -6 \end{bmatrix}$.

- As before, the black line is the eigenspace corresponding to the single eigenvector.
- The two green vectors are a basis for M_2.
- The red vector is $\mathbf{b}_3$, the end of the chain. It lies in M_3 but is not itself a member of M_2.
- The blue vector along the black line is the eigenvector $\mathbf{b}_1$. The gold vector is $\mathbf{b}_2$, a member of M_2 that is not in M_1. The blue and gold vectors replace the green ones as a basis for M_2.