Girding the Equator of the Earth with a Belt

Dr. Robert J. Lopez
Emeritus, RHIT
Maple Fellow, Maplesoft

Recently, I came across an addendum to a problem that appears in many calculus texts, an addendum I had never explored. It intrigued me, and I hope it will capture your attention too.

The problem is that of girding the equator of the earth with a belt, then extending by one unit (here, taken as the foot) the radius of the circle so formed. The question is by how much does the circumference of the belt increase. This problem usually appears in the section of the calculus text dealing with linear approximations by the differential. It turns out that the circumference of the enlarged band is 2π ft greater than the original band.

(An alternate version of this has the circumference of the band increased by one foot, with the radius then being increased by 0.16 ft.)

The addendum to the problem then asked how high would the enlarged band be over the surface of the earth if it were lifted at one point and drawn as tight as possible around the equator. At first, I didn't know what to think. Would the height be some surprisingly large number? And how would one go about calculating this height.

It turns out that the enlarged and lifted band would be some 616.67 feet above the surface of the earth! This is significantly larger than the increase in the diameter of the original band. So, the result is a surprise, at least to me. Here are the relevant calculations.

Assuming the equator of the earth is a perfect circle of radius 4000 miles, the circumference of a band around the equator is $C_0 = 2\pi \times 4000 \times 5280$ ft. If the radius is increased by one foot, the new circumference is $C_1 = 2\pi\,(4000 \times 5280 + 1)$, the difference being 2π.

Incidentally, if the band of circumference C_0 increases by one foot, then the radius increases by

$$\frac{C_0 + 1}{2\pi} - C_0 = \frac{1}{2\pi} \doteq 0.16 \text{ ft}$$

Figure 1 illustrates what happens if, at one point, the band of circumference C_1 is lifted as high as possible. The red arc, measured by angle θ, has length $\lambda = r\,\theta = r\,(\phi + \pi/2)$. The height of the lifted point above the surface of the earth is denoted by s, and the segment not in contact with the earth, tangent at the point where the band leaves the surface, has length t. The radius to this point of last contact is orthogonal to the tangent, and makes an angle ϕ with the horizontal.

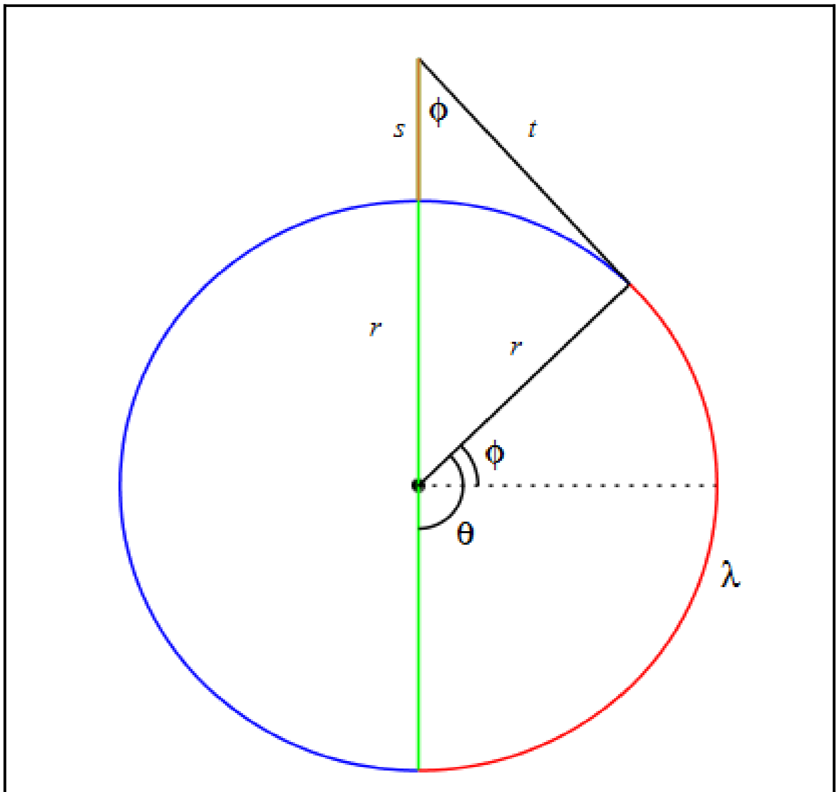

Figure 1 Band with circumference C_1 lifted at one point

As a consequence of symmetry and the right triangle in Figure 1, $\phi = \arcsin\left(\frac{r}{r+s}\right)$ and $t = \sqrt{(r+s)^2 - r^2} = \sqrt{s^2 + 2\,r\,s}$. Hence, we have

$$
\begin{aligned}
C_1/2 &= \lambda + t \\
&= r\,\theta + t \\
&= r\,(\pi/2 + \phi) + t \\
&= r\left(\pi/2 + \arcsin\left(\frac{r}{r+s}\right)\right) + \sqrt{s^2 + 2\,r\,s}
\end{aligned}
$$

from which it follows by numeric solution of the equation $\pi\,(r+1) = r\left(\pi/2 + \arcsin\left(\frac{r}{r+s}\right)\right) + \sqrt{s^2 + 2\,r\,s}$, that $s \doteq 616.67$. Indeed, using Maple, we obtain as the value of s (in feet)

$$s = fsolve\left(eval\left(\pi\,(r+1) = r\left(\pi/2 + \arcsin\left(\frac{r}{r+s}\right)\right) + \sqrt{s^2 + 2\,r\,s},\ r = 4000 \cdot 5280\right), s\right)$$

$$s = 616.6695885$$