The SIR model with births and deaths

Background

The classical SIR model assumes that a population can be divided into three distinct compartments (see e.g. Bloomfield[2009, pp 149]): S is the proportion of susceptibles, I is the proportion of infected persons and R is the proportion of persons that have recovered from infection and are now immune against the disease. This model was originally developed in 1927 by Kermack and McKendrik.

One extension to the classic SIR model is to add births and deaths to the model. Thus there is an inflow of new susceptibles and an outflow from all three compartments. In its most simple form, the birth rate is set equal to the death rate which is assumed not to be related to the infectious disease.
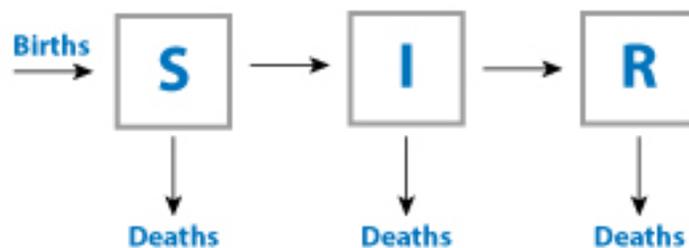

Figure 1

The with births and deaths extended SIR model can be described by three differential equations with three parameters:

$S(t)$: fraction of susceptibles

$I(t)$: fraction of infected persons

$R(t)$: fraction of recovered persons

$\mu \geq 0$: birth and death rate, $1/\mu$=life expectancy

$\beta \geq 0$: infection (transmission) rate

$\gamma \geq 0$: removal rate

(1) $$\frac{d}{dt}\left(S(t)\right) = \mu - \beta \cdot S(t) \cdot I(t) - \mu \cdot S(t)$$

(2) $$\frac{d}{dt}\left(I(t)\right) = \beta \cdot S(t) \cdot I(t) - \gamma \cdot I(t) - \mu \cdot I(t)$$

(3) $$\frac{d}{dt}\left(R(t)\right) = \gamma \cdot I(t) - \mu \cdot R(t)$$

Initial conditions:

$S(0) \in (0, 1]$

$I(0) \in (0, 1]$

$R(0) = 0$

$S(0) + I(0) = 1$

The basic reproduction ratio R_0 for the SIR model with as defined above can be derived as:

$$R_0 = \frac{\beta}{\gamma + \mu}$$

R_0 is the number of secondary infections that is produced by one primary infection in a wholly susceptible population. Only when R_0 > 1, an epidemic occurs. *1/(γ+μ)* is the average infectious period.

Setting equations (1)-(3) equal to zero and solving these for for the equilibrium solutions of $S(t)$, $I(t)$ and $R(t)$ in dependence of the parameters μ, β, and γ (e.g. by using the Maple procedure *solve()*) provides the following solution:

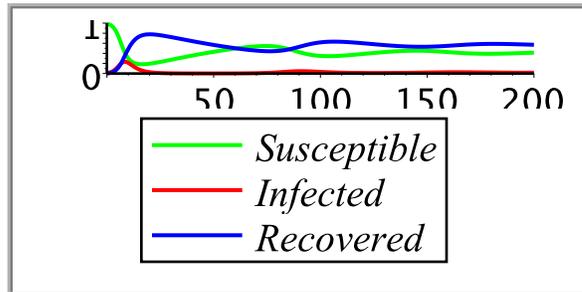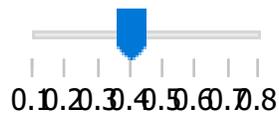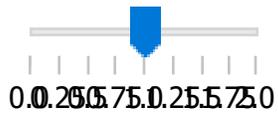|			
S_e	$=$	$\dfrac{\gamma + \mu}{\beta}$	$= \dfrac{1}{R_0}$
I_{eq}	$=$	$(\mu \cdot (\beta - \gamma - \mu)) / (\beta \cdot (\gamma + \mu))$	$= \dfrac{\mu}{\beta} \cdot (R_0 - 1)$
$R\backslash$	$=$	$(\gamma (\beta - \gamma - \mu)) / (\beta (\gamma + \mu))$	$= 1 - \dfrac{1}{R_0} - \dfrac{\mu}{\beta} \cdot (R_0 - 1)$

Warnings: (1) The basic reproduction ratio and the recovered persons use unfortunately the same capital letter R as symbol but one has to be aware that these are different entities. (2) Not all combinations of infection and removal rates within the ranges of the interactive model below will match a real existing infectious disease.

The SIR model exists in different flavors and parametrizations, the model as presented here follows Bloomfield[2009].
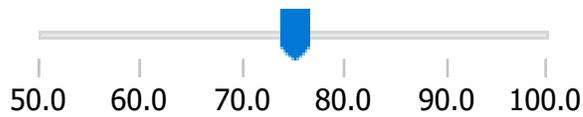
▼ Interactive SIR model

Infection	Removal rate (γ)

(transmission) rate (β)

0.0.250.50.750.1.251.51.752.0   0.10.20.30.40.50.60.70.8

- Susceptible
- Infected
- Recovered

| **Life expectancy (1/μ)** |

50.0   60.0   70.0   80.0   90.0   100.0

Model summary		
Parameter name	**Symbol**	**Value**
Susceptibles, fraction at t=0	$S(0)$	0.990
Infected, fraction at t=0	$I(0)$	0.010
Recovered, fraction at t=0	$R(0)$	0.000
Birth rate= Death rate	μ	0.013
Life expectancy	$\dfrac{1}{\mu}$	75.000

Infection (transmission) rate	β	1.000
Removal rate	γ	0.400
Basic reproduction ratio	$R_0 = \dfrac{\beta}{\gamma + \mu}$	2.419
S_{eq}	$\dfrac{1}{R_0}$	0.413
I_{eq}	$\dfrac{\mu}{\beta} \cdot (R_0 - 1)$	0.018
R_{eq}	$1 - S_{eq} - I_{eq}$	0.567

▼ References

Bloomfield V [2009]: *Computer Simulation and Data Analysis in Molecular Biology and Biophysics. An Introduction Using R.* Springer, Dordrecht, Heidelberg, London, NewYork

Kermack WO, McKendrick AG [1927]: *A contribution to the Mathematical Theory of Epidemics.* Proc. Roy. Soc. A 115, 700-711 (Download link)