Application Demonstration
www.maplesoft.com

Classroom Tips and Techniques: Drawing a Normal and Tangent Plane on a Surface

Robert J. Lopez
Emeritus Professor of Mathematics and Maple Fellow
Maplesoft

Introduction

A question posted to MaplePrimes on May 25, 2013, asked for help in drawing a normal and tangent plane on a given surface. Posted on the Sunday prior to the American Memorial Day holiday, it was probably a request for help on a homework assignment. As such, it went unanswered, but it's such a standard problem in multivariate calculus, that it begs for a solution. In fact, there are at least four different ways to solve the problem in Maple, and this article discusses them all.

The Problem

The actual post to MaplePrimes can be found here. In essence, it asks for the normal and tangent plane at $(2, 1, 3)$ on the surface defined by $z = f(x, y) \equiv 10 - x^2 - 3\,y^2$.

Solution 1

Table 1 contains a solution based on the new "lines and planes" commands recently added to the Student *MultivariateCalculus* package in Maple 17. The solution is implemented via explicit calls to commands, but the calculation could equally well be done in a syntax-free way through the Context Menu.

Initializations	
• Click the restart icon in the toolbar, or execute the **restart** command at the right.	*restart*
• Tools>Load Package: Student Multivariate Calculus	Loading Student:-MultivariateCalculus
• Assign to the name *f*, the expression for $z(x, y)$.	$f := 10 - x^2 - 3\,y^2:$
• Define *P*, the point of contact.	$P := [2, 1, 3]:$
Obtain a normal vector at the point *P*	
• Apply the **Gradient** command to the	$\mathbf{N} := Gradient(z-f, [x, y, z] = P)[\ ]$

expression $z - f$. The **Gradient** command in the *MultivariateCalculus* package has simple syntax for evaluating the gradient vector at a point.	$\begin{bmatrix} 4 \\ 6 \\ 1 \end{bmatrix}$
Obtain the tangent plane	
• Invoke the **Plane** command with the point P and normal vector **N** as arguments. • Display the equation of the plane with the **GetRepresentation** command.	$T := Plane(\mathbf{N}, P)$: $GetRepresentation(T)$ $4x + 6y + z = 17$

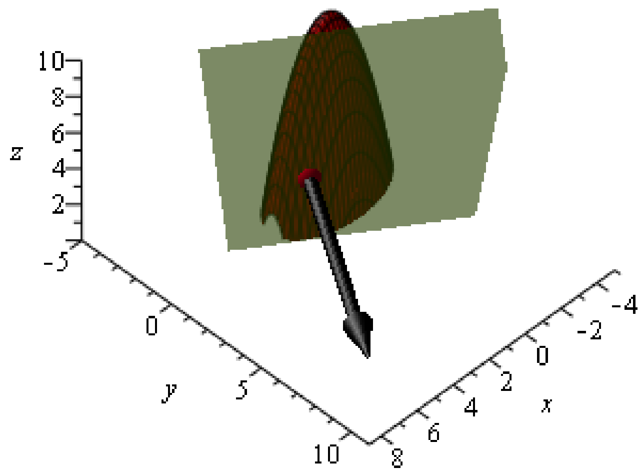
Table 1 Tangent plane and normal obtained in the Student *MultivariateCalculus* package

Figure 1 shows the surface, the tangent plane, and the normal vector. The surface is drawn with the **plot3d** command; the tangent plane, normal, and point of contact, with the **GetPlot** command in the *MultivariateCalculus* package. The two separate images are joined with the **display** command in the *plots* package.

$p_1 := plot3d(f, x = -3..3, y = -3..3, color = red)$:

$p_2 := GetPlot(T, planeoptions = [transparency = .4], normaloptions = [color = black])$:

$plots\text{:-}display(p_1, p_2, view = 0..10, axes = frame, labels = [x, y, z])$



The plane $4x+6y+z=17$. A point on the plane. The normal vector.

Figure 1 Surface drawn by **plot3d**; tangent plane and normal, by **GetPlot**

Solution 2

Table 2 contains a solution based on commands in the Student *VectorCalculus* package. As in Table 1, explicit commands are employed; the calculations also can be implemented in a syntax-free way through the Context Menu.

Initializations	
• Click the restart icon in the toolbar, or execute the **restart** command at the right.	*restart*
• Tools>Load Package: Student Vector Calculus	Loading Student:-VectorCalculus
• Assign to the name *f*, the expression for $z(x, y)$.	$f := 10 - x^2 - 3y^2$:
• Define *P*, the point of contact.	$P := [2, 1, 3]$:

Obtain a normal vector at the point P	
• The **Gradient** command in the *VectorCalculus* packages returns a **VectorField** object. Evaluation at a point via the **evalVF** command returns a **RootedVector** that "knows" the location of its root point, or "tail".	$\mathbf{N} := evalVF(Gradient(z-f), \langle P \rangle)$ $\begin{bmatrix} 4 \\ 6 \\ 1 \end{bmatrix}$
Obtain the equation of the tangent plane	
• In the *VectorCalculus* packages, the **TangentPlane** command returns the parametric form of a tangent plane, placing the component expressions in a vector.	$Z := TangentPlane(f, x=2, y=1)[3]:$ $z = Z$ $z = -4x + 17 - 6y$

Table 2 Tangent plane and normal obtained in the Student *VectorCalculus* package

Figure 2 shows the surface, the tangent plane, and the normal vector. The surface is drawn with the **plot3d** command; the tangent plane, with the **implicitplot3d** command; and the normal vector, with the **PlotVector** command in the Student *VectorCalculus* package.

$p_1 := plot3d(f, x=-3..3, y=-3..3, color=red):$

$p_2 := PlotVector(\mathbf{N}, color=black):$

$p_3 := plots\text{:-}implicitplot3d(z=Z, x=1..3, y=-3..3, z=1..5, color=cyan, style=surface):$

$plots\text{:-}display\left(p_1, p_2, p_3, axes=frame, scaling=constrained, view=0..10, labels=[x, y, z], orientation=[25, 80, 0]\right)$

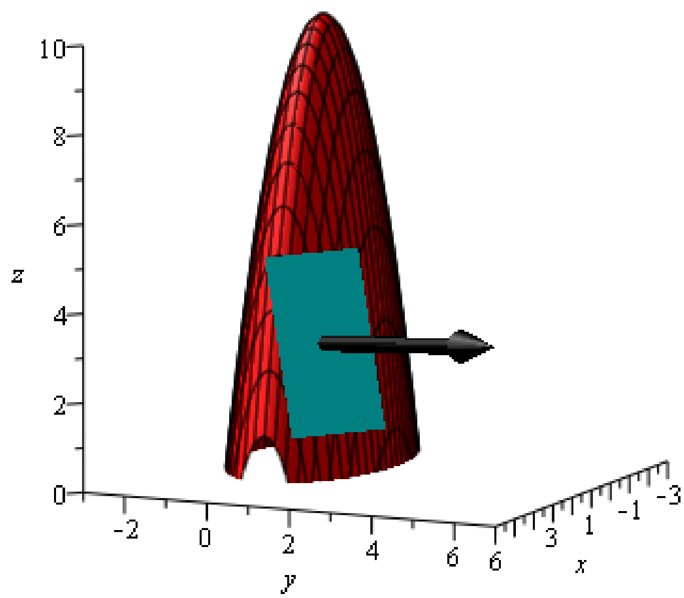


Figure 2 Surface, tangent plane, and normal vector

Each component in Figure 2 is drawn with a different command. The surface is drawn with the **plot3d** command that resides at top level. The tangent plane is drawn with the **implicitplot3d** command in the *plots* package. The advantage here is the greater control over the size of the tangent plane. Finally, the normal vector is drawn with the **PlotVector** command in the Student *VectorCalculus* package. This command respects the root point when graphing a **RootedVector**.

Solution 3

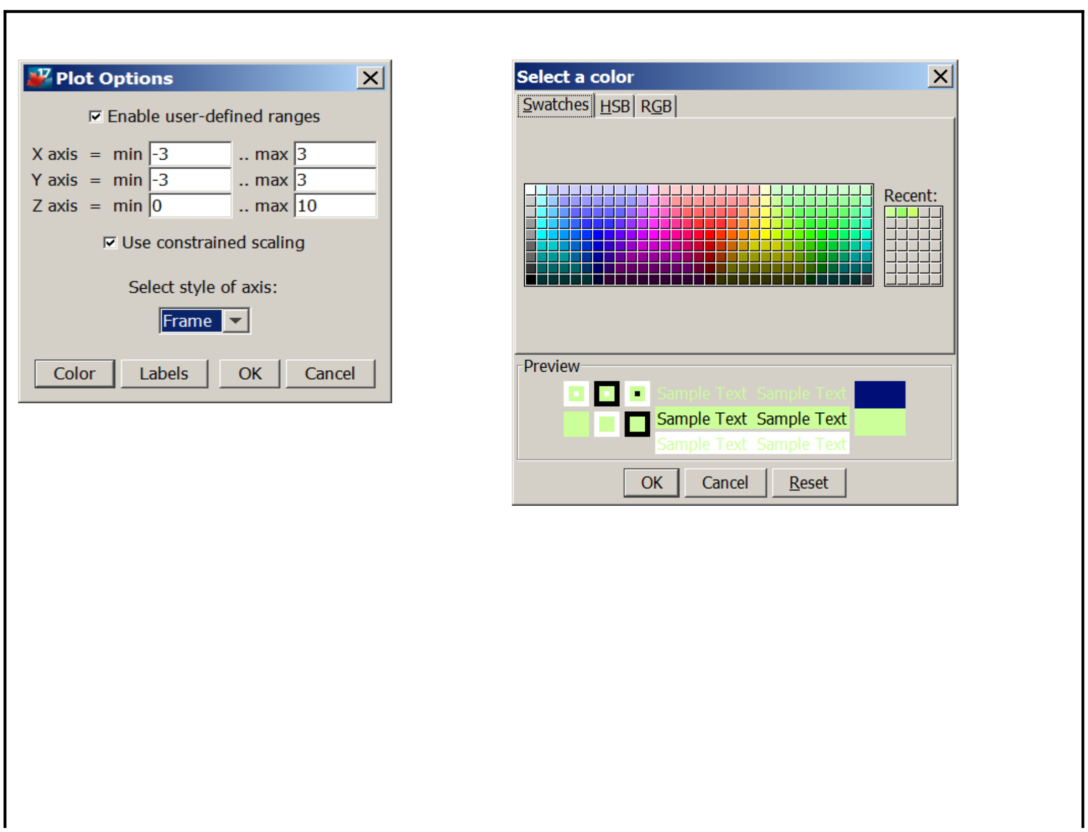
A third solution obtains the tangent plane as the degree-one Taylor polynomial approximation to *f*. This approximation can be obtained via the **TaylorApproximation** command, or via the Taylor Approximation tutor. Table 3 first initializes the calculation; the interaction with the tutor is captured by the static image in Figure 3.

Initializations	
• Click the restart icon in the toolbar, or execute the **restart** command at the right.	*restart*

• Tools>Load Package: Student Multivariate Calculus	Loading Student:-MultivariateCalculus
• Assign to the name f, the expression for $z(x, y)$.	$f := 10 - x^2 - 3\,y^2$:
• Define P, the point of contact.	$P := [2, 1, 3]$:
• As per Table 1, apply the **Gradient** command to the expression $z - f$, again making use of the simple syntax for evaluating the gradient vector at a point.	$\mathbf{N} := Gradient(z - f, [x, y, z] = P)[\;]$ $\begin{bmatrix} 4 \\ 6 \\ 1 \end{bmatrix}$
• Type f Context Menu: Evaluate and Display Inline • Context Menu: Tutors>Calculus-Multivariate>Taylor Approximation • Adjust the tutor as per Figures 3 and 4, below.	$f = -x^2 - 3\,y^2 + 10 \xrightarrow{\text{Taylor approximation tutor}}$

Table 3 Solution via Taylor Approximation tutor

As per Figure 4, set the coordinates for the point of contact, and change the default degree (which is 5) to 1. Click the Plot Options button to obtain the dialogs shown Figure 3. Change the size of the bounding box for the graph, apply constrained scaling, and change the axes to "frame." Click the Color button to select color change for the Taylor series, and for the tangent plane, select a color lighter than the default color.



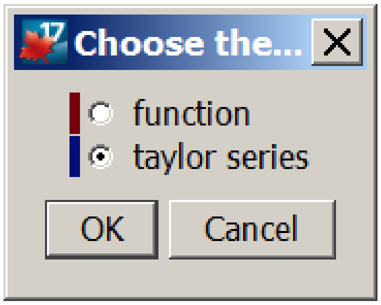


Figure 3 Auxiliary dialogs associated with the Taylor Approximation tutor

The graph in the tutor is returned to the worksheet by clicking the Close button. Note the TaylorApproximation command at the bottom of the tutor. It can be copied and pasted, thereby obtaining the graph directly, without going through the intermediary of the tutor.

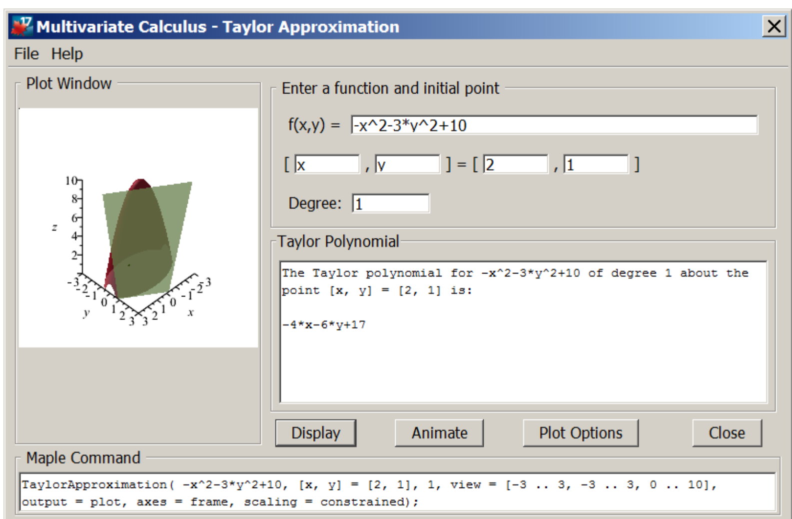


Figure 4 Tangent plane obtained as the degree-one Taylor polynomial approximation

Figure 6 is obtained from the graph returned by the tutor by adding the normal vector via a drag-and-drop (or copy/paste) of the image of the vector in Figure 5.

$plots\text{:-}arrow(P, N, color = black)$

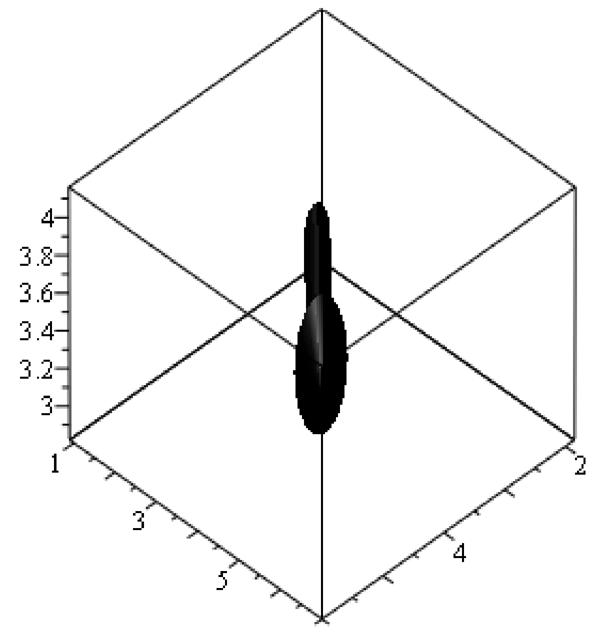


Figure 5 Normal vector drawn with the **arrow** command in the *plots* package

After interactively adjusting the axes via the Context Menu for the graph of the surface and tangent plane, and adding the graph of the vector, the result is Figure 6, comparable to Figures 1 and 2.

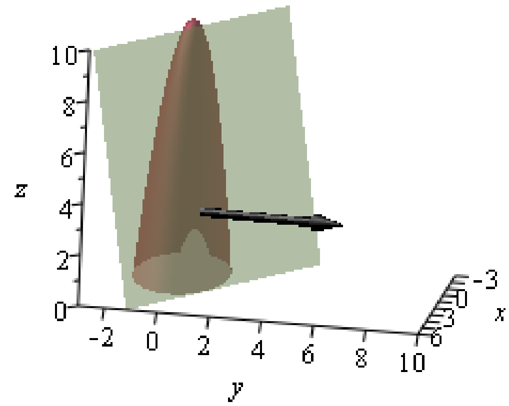


Figure 6 Graph of surface, tangent plane and normal vector

Of course, the graph returned by the tutor can be generated directly, as per Figure 7, where it is assigned to a name so that it can be used in the **display** command that generates Figure 8.

$p_1 := TaylorApproximation(f, [x, y] = [2, 1], 1, output = plot, axes = frame, view = [-3..6, -3..10, 0..10], caption = "", tayloroptions = [color = cyan]) :$

p_1

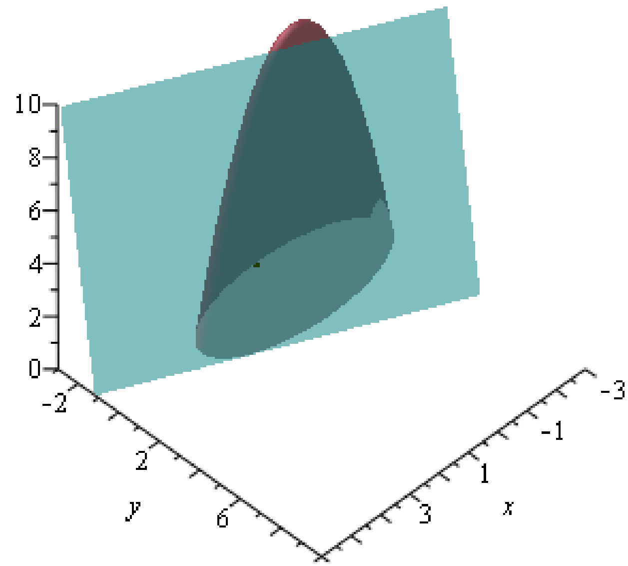


Figure 7 Graph of surface and tangent plane generated by the **TaylorApproximation** command

Figure 8 combines the graph in Figure 7 with a graph of the normal vector.

$plots{:}\text{-}display\left(p_1, plots{:}\text{-}arrow(P, N, color = black), scaling = constrained\right)$

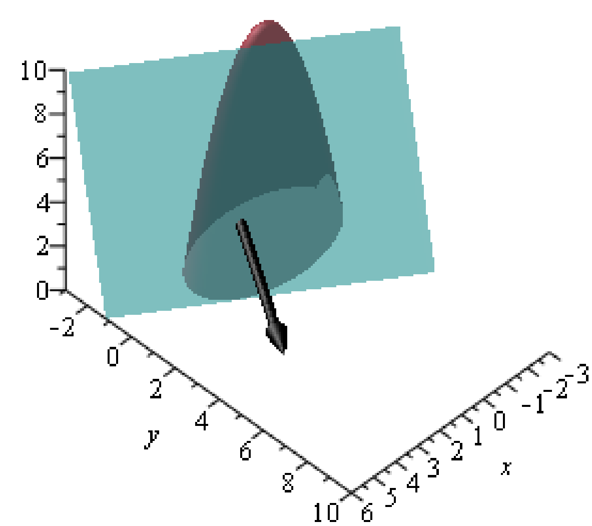


Figure 8 Alternate method for obtaining the graph in Figures 1, 2, and 6

Solution 4

Table 4 contains top-level calculations for the normal vector and the equation of the tangent plane. No special packages are therefore used. In place of the partial derivative templates taken from the Expression palette, the **diff** command could have been used. Also, in place of the Evaluation template, the **eval** command could have been used.

The dot product operator from the Common Symbols palette could have been replaced with the period. The vector form of the equation of a plane is compact. The algebraic alternative of writing $4x + 6y + z = d$, and determining d by the substitutions $(x, y, z) = P$, would require additional steps.

Initialize	
• Click the restart icon in the toolbar, or execute the **restart** command at the right.	*restart*
• Assign to the name *f*, the expression for $z(x, y)$.	$f := 10 - x^2 - 3y^2 :$
• Define the point of contact as the vector **P**.	$\mathbf{P} := \langle 2, 1, 3 \rangle :$

Obtain a normal vector at P	
• Obtain the vector $\mathbf{v} = \begin{bmatrix} -f_x \\ -f_y \\ 1 \end{bmatrix}$, a normal to $z=f(x,y)$. • Expression palette: Evaluation template Evaluate **v** at the point of contact.	$\mathbf{N} := \left\langle -\frac{\partial}{\partial x} f, -\frac{\partial}{\partial y} f, 1 \right\rangle \Big\vert_{x=2,\, y=1} = \begin{bmatrix} 4 \\ 6 \\ 1 \end{bmatrix}$
Obtain the equation of the tangent plane	
• Using the vector form of the equation of a plane containing the point P, and having normal **N**, obtain Q, the required tangent plane.	$Q := (\langle x, y, z\rangle - \mathbf{P}) \cdot \mathbf{N} = 0$ assuming *real* $4x - 17 + 6y + z = 0$
Table 4 Top-level calculation of a normal and the equation of the tangent plane	

Figure 9, comparable to Figures 1, 2, 6, and 8, is obtained from separate graphs of the surface, tangent plane, and normal vector.

$p_1 := plot3d(f, x=-3..3, y=-3..3):$

$p_2 := plots:\text{-}implicitplot3d(Q, x=1..3, y=0..2, z=2..4, style=surface, color=cyan):$

$p_3 := plots:\text{-}arrow(P, N, color=black):$

$plots:\text{-}display(p_1, p_2, p_3, scaling=constrained, axes=frame, view=0..10)$

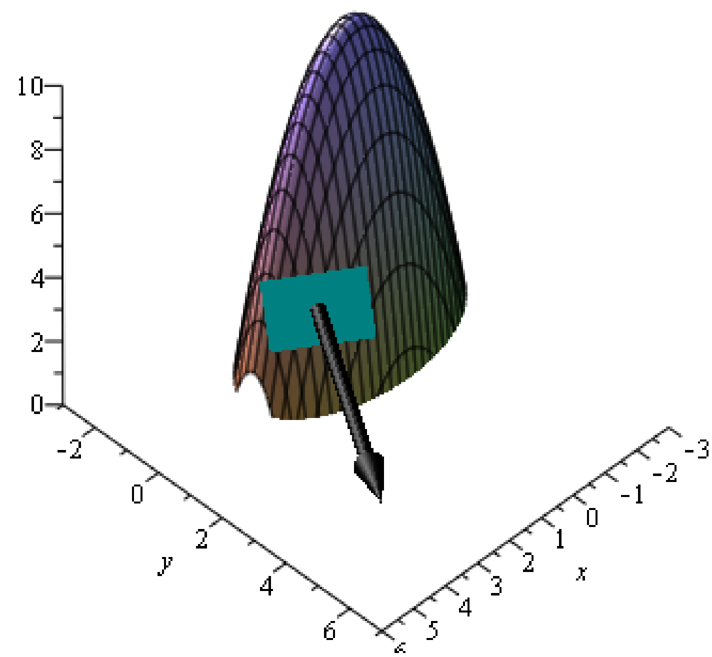


Figure 9 Graph of surface, tangent plane, and normal vector