Classroom Tips and Techniques:

Norm of a Matrix

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Introduction

The greatest benefits from bringing Maple into the classroom are realized when the static pedagogy of a printed textbook is enlivened by the interplay of symbolic, graphic, and numeric calculations made possible by technology. It is not enough merely to compute or check answers with Maple. To stop after noting that indeed, Maple can compute the correct answer is not a pedagogical breakthrough.

Getting Maple to compute the correct answer is just the first step. Using Maple to bring insights not easily realized with by-hand calculations should be the goal of everyone who sets a hand to improving the learning experiences of students.

For example, let's look at how the notion of a matrix norm might be taught in a Maple environment. In particular, let's consider the definition

$$\| A \| = \max\{ \| A x \| \}, \ \| x \| = 1$$

where the vector norm $\| A \mathbf{x} \|$ is the Euclidean, or 2-norm, so that $\| A \|$ is actually $\| A \|_2$, the 2-norm of the matrix A.

First we will show that Maple easily computes $\| A \|_2$, then we'll show how Maple can be used to gain some insight into just what this number means.

Initialize

• Tools>Load Package: Student Calculus 1	Loading Student:-Calculus1
• Tools>Load Package: Student Multivariate Calculus	Loading Student:-MultivariateCalculus
• Matrix palette: 2×2 matrix template Type entries and use Tab key to move through fields. • Context Menu: Assign to a Name>*A*	$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{\text{assign to a name}} A$
• Matrix palette: 2×1 matrix template for vector Type entries and use Tab key to move through fields. • Context Menu: Assign to a Name>**U**	$\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} \xrightarrow{\text{assign to a name}} U$

Preliminary Views

In Figure 1, the surface represents the Euclidean norm of $A\mathbf{V}$, that is,

$$\|A\mathbf{V}\| = \sqrt{10\,x^2 + 28\,x\,y + 20\,y^2}$$

where $\mathbf{V} = x\,\mathbf{i} + y\,\mathbf{j}$. The black curve drawn in the surface is the intersection of the surface with the lift of the unit circle up to the surface. The norm of A is the maximum of $\|A\mathbf{V}\|$ on the unit circle, that is, along the black curve traced in the surface.

In Figure 2, the slider controls the value of θ for the unit vector $\mathbf{u} = \cos(\theta)\;\mathbf{i} + \sin(\theta)\;\mathbf{i}$. As **u** rotates around the unit circle, the green vector, $A\mathbf{u}$, grows and shrinks. The length of the vector $A\mathbf{u}$ is displayed beneath the slider. The norm of A itself is the length of the longest green vector. In other words, Figure 2 demonstrates that the norm of a matrix is the largest factor by which the length of a unit vector can be changed by the matrix.

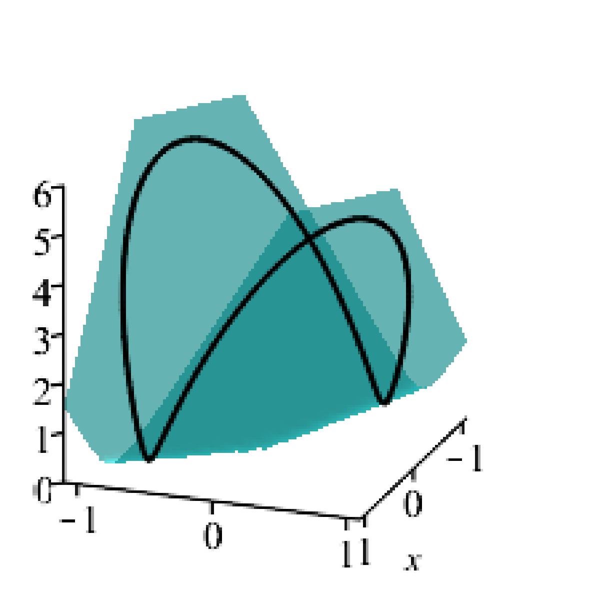


Figure 1 Norm of $A\mathbf{V}$ and the lift of the unit circle

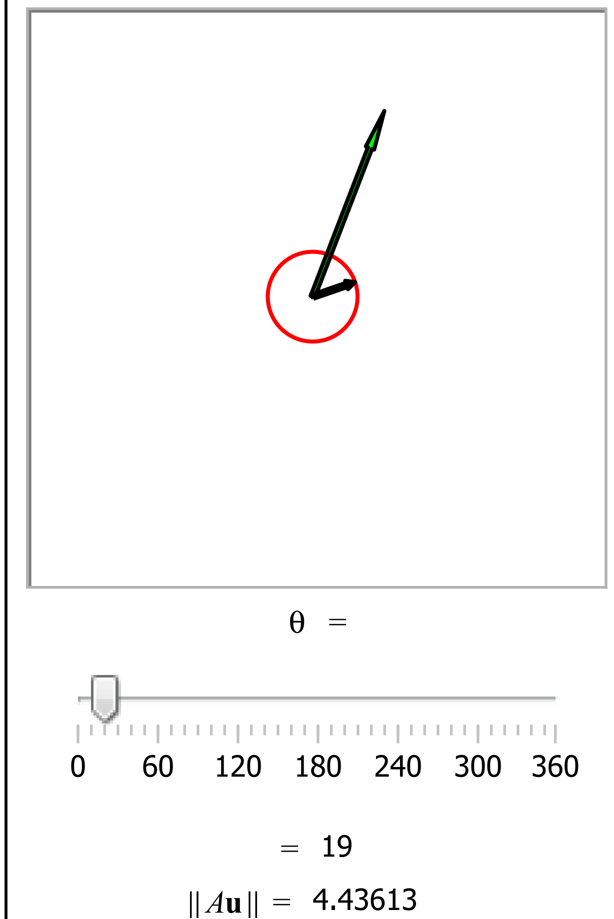


Figure 2 Unit vector **u** (in black); $A\mathbf{u}$ (in green)

Obtain the Euclidean Norm of A

Table 1 demonstrates how to obtain the Euclidean norm of A by syntax-free methods in Maple. The chief tool is the Context Menu system.

• Type A, the name of the matrix. Context Menu: Evaluate and Display Inline • Context Menu: Norm>Euclidean • Context Menu: Approximate>10 (digits)
$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{\text{Euclidean-norm}} \sqrt{15+\sqrt{221}} \xrightarrow{\text{at 10 digits}} 5.464985704$
Table 1 Syntax-free calculation of $\|A\|$, the Euclidean norm of A

Matrix Norm from First Principles

In Table 2, $\|A\|$, the Euclidean norm of A, is obtained via first principles.

For the product $A\mathbf{U}$, obtain $\|A\mathbf{U}\|$, its Euclidean norm, and a graph of $\|A\mathbf{U}\|$
• Write the product of A and $\mathbf{U}$, using a period for noncommutative multiplication; press the Enter key. • Context Menu: Student Multivariate Calculus≻Norm • Context Menu: Simplify≻Simplify • Context Menu: Plots≻Plot Builder (Set $t \in [0, 2\pi]$.)

$A.\mathbf{U}$

$$\begin{bmatrix} \cos(t) + 2\sin(t) \\ 3\cos(t) + 4\sin(t) \end{bmatrix} \quad (1)$$

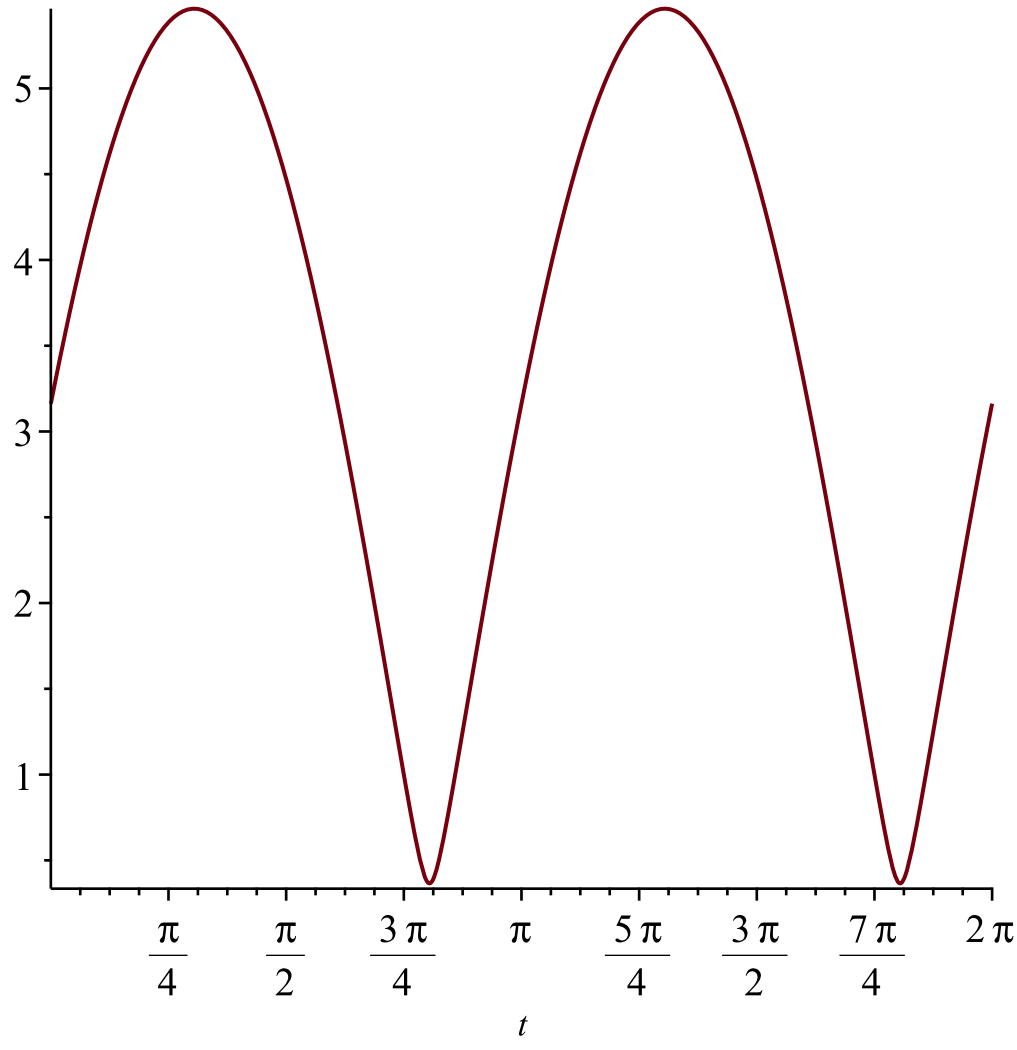
$\xrightarrow{\text{norm}}$

$$\sqrt{(\cos(t) + 2\sin(t))^2 + (3\cos(t) + 4\sin(t))^2} \quad (2)$$

$\overset{\text{simplify}}{=}$

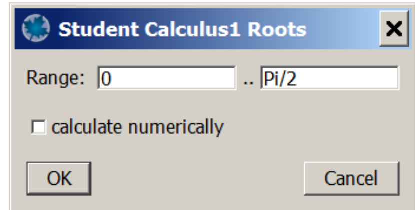
$$\sqrt{-10\cos(t)^2 + 28\cos(t)\sin(t) + 20} \quad (3)$$

$\rightarrow$



Set the derivative of $\|A\mathbf{U}\|$ to zero and solve for t_{max}

- Calculus palette: Differentiation operator
  Reference $\|A\mathbf{U}\|$ by its equation label
- Context Menu: Student Calculus 1 > Solve > Find Roots
  Complete Roots dialog as per figure to the right.
- Context Menu:Select Element > 1



$$\frac{d}{dt}\,\mathbf{(3)} = 0$$

$$\frac{1}{2}\,\frac{20\cos(t)\sin(t) - 28\sin(t)^2 + 28\cos(t)^2}{\sqrt{-10\cos(t)^2 + 28\cos(t)\sin(t) + 20}} = 0 \qquad \mathbf{(4)}$$

$\xrightarrow{\text{roots}}$

$$\left[\arctan\left(\frac{5}{14} + \frac{1}{14}\sqrt{221}\right)\right] \qquad \mathbf{(5)}$$

$\xrightarrow{\text{select entry 1}}$

$$\arctan\left(\frac{5}{14} + \frac{1}{14}\sqrt{221}\right) \qquad \mathbf{(6)}$$

Substitute t_{max} into $\|A\mathbf{U}\|$ to obtain $\|A\|$

- Expression palette: Evaluation template (Reference $\|AU\|$ and t_{max} by equation label.)
  Context Menu: Label > Label Reference
- Press the Enter key.
- Context Menu: Simplify > Simplify
- Context Menu: Approximate > 10 (digits)

$$\left.\sqrt{-10\cos(t)^2 + 28\cos(t)\sin(t) + 20}\,\right|_{t = \arctan\left(\frac{5}{14} + \frac{1}{14}\sqrt{221}\right)}$$

$$\sqrt{-\frac{10}{1 + \left(\frac{5}{14} + \frac{1}{14}\sqrt{221}\right)^2} + \frac{28\left(\frac{5}{14} + \frac{1}{14}\sqrt{221}\right)}{1 + \left(\frac{5}{14} + \frac{1}{14}\sqrt{221}\right)^2} + 20} \qquad \mathbf{(7)}$$

$\overset{\text{simplify}}{=}$

$$\frac{2\sqrt{1105 + 74\sqrt{221}}}{\sqrt{221 + 5\sqrt{221}}} \qquad \mathbf{(8)}$$

$\xrightarrow{\text{at 10 digits}}$

$$5.464985704 \qquad \mathbf{(9)}$$

Table 2 Calculation of $\|A\|$ from first principles

Alternate Approaches

An alternate approach treats the calculation of $\|A\|$ as a constrained optimization problem, where the objective function is $\|A\mathbf{V}\|$ and the constraint is the equation $\|\mathbf{V}\| = 1$. Table 3 uses the Context Menu to implement a numeric optimization based on the **Maximize** command in the *Optimization* package.

• Write the sequence of objective function and constraint equation, then press the Enter key. • Context Menu: Optimization>Maximize (local)
$\|A.\langle x, y\rangle\|, x^2 + y^2 = 1$ $\sqrt{(x+2y)^2 + (3x+4y)^2}, x^2 + y^2 = 1$ $\xrightarrow{\text{maximize}}$ $[5.46498570421913055, [x = 0.576048436922496, y = 0.817415560360323]]$
Table 3 Numeric solution of a constrained optimization problem

The first number in the returned list is the maximum of the objective function, hence, the norm of the matrix A. The second member of that list is a list of equations specifying x_{max} and y_{max}.

The symbolic equivalent of the numeric calculation in Table 3 is realized in Table 4 where the Lagrange Multiplier method is implemented. It is surprisingly "messy," especially since there are multiple solutions that have to be sorted out.

Obtain x_{max} and y_{max} via the Lagrange Multiplier method
• Write the function $f + \lambda g$, where f is the objective function and g is the constraint; press the Enter key. • Calculus palette: Partial-derivative operator (Differentiate with respect to x, y, and λ; press the Enter key.) • Context Menu: Solve>Solve (explicit)

$$\|A.\langle x,y\rangle\| + \lambda\,(x^2+y^2-1)$$

$$\sqrt{(x+2y)^2+(3x+4y)^2} + \lambda\,(x^2+y^2-1) \qquad \textbf{(10)}$$

$$\frac{\partial}{\partial x}\,\textbf{(10)} = 0,\ \frac{\partial}{\partial y}\,\textbf{(10)} = 0,\ \frac{\partial}{\partial \lambda}\,\textbf{(10)} = 0$$

$$\frac{1}{2}\,\frac{20x+28y}{\sqrt{(x+2y)^2+(3x+4y)^2}} + 2\lambda x = 0,\ \frac{1}{2}\,\frac{28x+40y}{\sqrt{(x+2y)^2+(3x+4y)^2}} + 2\lambda y = 0,\ x^2+y^2-1=0 \qquad \textbf{(11)}$$

$$\xrightarrow{\text{solve}}$$

$$\left\{\lambda = -\frac{1}{4}\sqrt{34}+\frac{1}{4}\sqrt{26},\ x = \frac{1}{442}\sqrt{97682+1105\sqrt{34}\sqrt{26}},\ y = \frac{1}{3094}\sqrt{97682+1105\sqrt{34}\sqrt{26}}\left(2\left(-\frac{1}{4}\sqrt{34}+\frac{1}{4}\sqrt{26}\right)^2-5\right)\right\},\ \left\{\lambda = -\frac{1}{4}\sqrt{34}+\frac{1}{4}\sqrt{26},\ x = -\frac{1}{442}\sqrt{97682+1105\sqrt{34}\sqrt{26}},\ y = -\frac{1}{3094}\sqrt{97682+1105\sqrt{34}\sqrt{26}}\left(2\left(-\frac{1}{4}\sqrt{34}+\frac{1}{4}\sqrt{26}\right)^2-5\right)\right\},\ \left\{\lambda = -\frac{1}{4}\sqrt{34}-\frac{1}{4}\sqrt{26},\ x = \frac{1}{442}\sqrt{97682-1105\sqrt{34}\sqrt{26}},\ y = \frac{1}{3094}\sqrt{97682-1105\sqrt{34}\sqrt{26}}\left(2\left(-\frac{1}{4}\sqrt{34}-\frac{1}{4}\sqrt{26}\right)^2-5\right)\right\},\ \left\{\lambda = -\frac{1}{4}\sqrt{34}-\frac{1}{4}\sqrt{26},\ x = -\frac{1}{442}\sqrt{97682-1105\sqrt{34}\sqrt{26}},\ y = -\frac{1}{3094}\sqrt{97682-1105\sqrt{34}\sqrt{26}}\left(2\left(-\frac{1}{4}\sqrt{34}-\frac{1}{4}\sqrt{26}\right)^2-5\right)\right\} \qquad \textbf{(12)}$$

Substitute x_{max} and y_{max} into $\|A.\langle x,y\rangle\|$ to determine $\|A\|$

- Expression palette: Evaluation template (Evaluate $\|A.\langle x,y\rangle\|$ at the third solution and press the Enter key.
- Context Menu: Simplify≻Simplify
- Context Menu:≻Approximate≻10 (digits)

$$\| A.\langle x, y\rangle \| \Big|_{(12)[3]}$$

$$\left(\left(\frac{1}{442}\sqrt{97682-1105\sqrt{34}\sqrt{26}}+\frac{1}{1547}\sqrt{97682-1105\sqrt{34}\sqrt{26}}\left(2\left(-\frac{1}{4}\sqrt{34}-\frac{1}{4}\sqrt{26}\right)^2-5\right)\right)^2+\left(\frac{3}{442}\sqrt{97682-1105\sqrt{34}\sqrt{26}}+\frac{2}{1547}\sqrt{97682-1105\sqrt{34}\sqrt{26}}\left(2\left(-\frac{1}{4}\sqrt{34}-\frac{1}{4}\sqrt{26}\right)^2-5\right)\right)^2\right)^{1/2} \quad (13)$$

$$\overset{\text{simplify}}{=}$$

$$\sqrt{15+\sqrt{17}\sqrt{13}} \quad (14)$$

$$\xrightarrow{\text{at 10 digits}}$$

$$5.464985704 \quad (15)$$

Table 4 Symbolic calculation of $\| A \|$ by the Lagrange Multiplier method