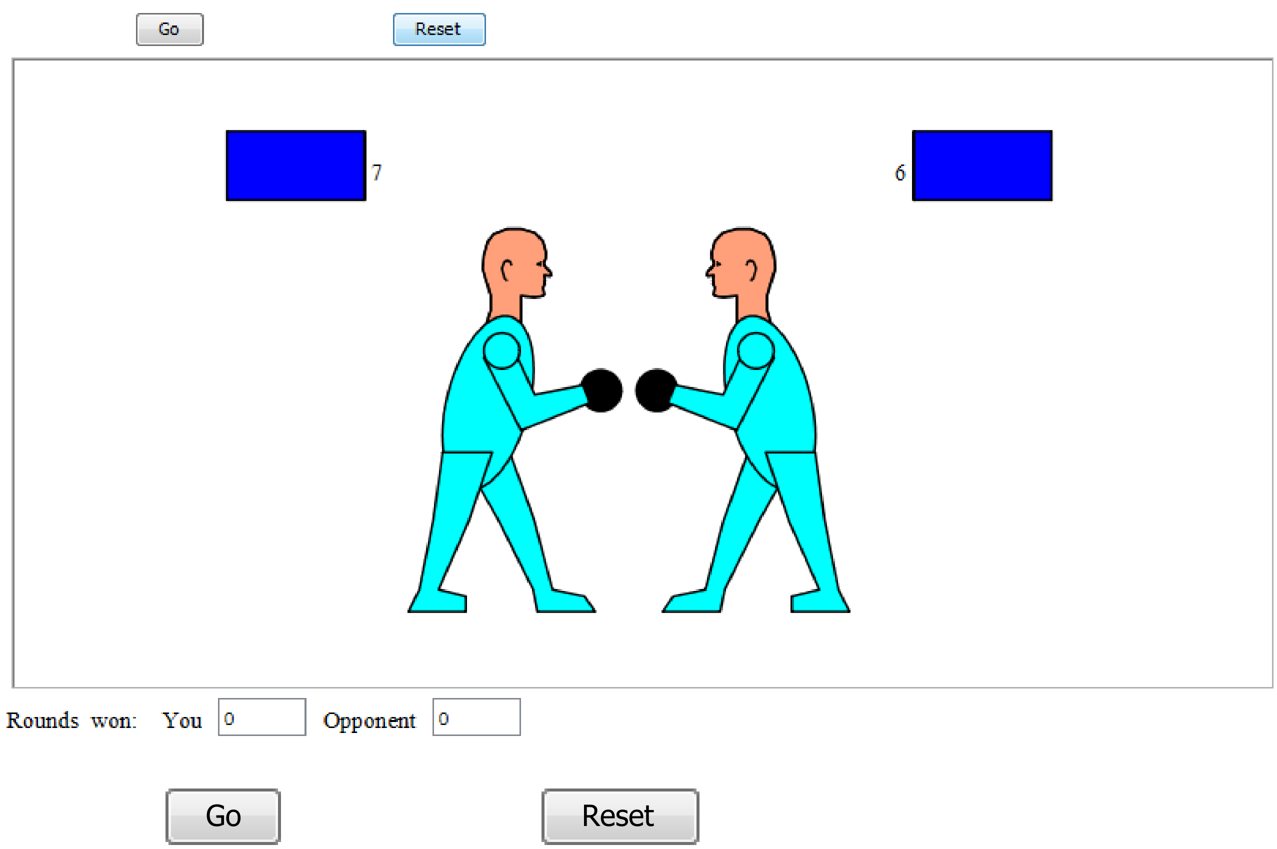
Street-Fighting Maple

Fighting games such as the Street Fighter® series are very popular. If you haven't played them yourself, you've seen your friends (or perhaps your children) playing them. But you may not be aware of the mathematical side of such games.

In the simplest form of such a game, your character and an opponent are fighting against each other. Each starts out with a certain number of **hit points**, which may be displayed in a **life bar**.As the game progresses, whenever one player succeeds in hitting the other the player who was hit loses a certain number of hit points. Whenever a player's hit points are reduced to 0, that player loses this round of the game and the other player wins. Thus the objective of the game is to reduce your opponent's hit points to 0 before your own reaches 0. There may be several rounds in the game, but we'll consider just a single round.

A simple fight

Here is a rather crude Maple implementation of a simple fighting game. Your character is the one on the left. There's no strategy involved here: just press the "Go" button to bring the fighters together. Each time you press "Go", one fighter will hit the other, and the one hit will lose 1 hit point. The "Reset" button will restore both fighters to their original hit points.



The initial numbers of hit points, and the probabilities of a hit for each fighter, can be changed below (**do not** press Enter: just click somewhere out of the text area when done). Note that the two probabilities will always add up to 1. You can also give names to the two fighters.

	You	Opponent
Initial hit points	7	6
Probability of scoring a hit	2/5	3/5

Winning probabilities

The mathematical question here is: given the initial hit points of the two fighters and their probabilities of scoring a hit, what is the probability that your fighter wins the round?

The answer can be based on the binomial distribution, which says that in n independent trials with probability p of success in each trial, the probability of having exactly x successes is $\binom{n}{x} p^x (1-p)^{n-x}$ for any integer x from 0 to n.

Suppose you have h_1 hit points and your opponent has h_2 hit points. Imagine that instead of stopping when one fighter's hit points reach 0, the round continued for a total of $n = h_1 + h_2 - 1$ hits. At the end, the sum of the hit points of the two fighters will be 1: the loser will have 0 or negative hit points, the winner will still have at least one hit point. So we can tell from this extended version of the round which fighter wins. The probability that you are the winner, i.e. that you have scored at least h_2 hits, is thus obtained by adding up the probability of scoring x hits in the n rounds for x from h_2 to n.

$$W(h_1, h_2) = \sum_{x=h_2}^{h_1+h_2-1} \binom{h_1+h_2-1}{x} p^x (1-p)^{h_1+h_2-1-x}$$

Here's another way of calculating the result, without imagining continuing the round. In order for you to win with exactly r hit points left, there must be a total of $h_1 - r + h_2$ hits; out of the first $h_1 - r + h_2 - 1$ hits you score $h_2 - 1$, and then you score the last hit to finish off your opponent. Since the first $h_1 - r + h_2 - 1$ hits and the $h_1 - r + h_2$ th hit are independent, the probability that you do this is

$$\binom{h_1 - r + h_2 - 1}{h_2 - 1} p^{h_2 - 1} (1-p)^{h_1 - r} p = \binom{h_1 - r + h_2 - 1}{h_2 - 1} p^{h_2} (1-p)^{h_1 - r}$$

Thus the total probability that you win the round can be obtained by adding these up for r from 1 to h_1:

$$W(h_1, h_2) = \sum_{r=1}^{h_1} \binom{h_1 - r + h_2 - 1}{h_2 - 1} p^{h_2} (1-p)^{h_1 - r}$$

You could consider this as $F(h_1 + h_2 - 1)$ where F is the cumulative distribution function for a Negative Binomial random variable with parameters h_2 and p. This is the distribution for the number of trials until the h_2'th success with probability p of success in each one.

These probabilities are not only valid at the beginning of the round: at any time during the round, given that you now have h_1 hit points left and your opponent has h_2, your probability of winning the round is given by the formula above.

The following table gives your probability $W(h_1, h_2)$ of winning, based on your probability p of scoring a hit, for each pair (h_1, h_2) up to 6.

Your probability of scoring a hit	.5

Opponent's Hit Po...

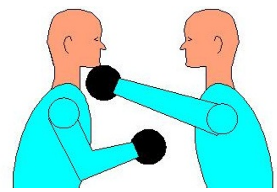
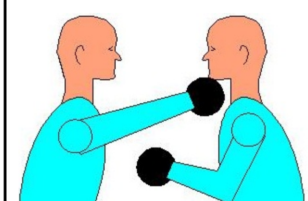
Your Hit Points	1	2	3	4	5	6
5	`0.9688`	`0.8906`	`0.7734`	`0.6367`	`0.5000`	`0.3770`
6	`0.9844`	`0.9375`	`0.8555`	`0.7461`	`0.6230`	`0.5000`
7	0	0	0	0	0	0
8	0	0	0	0	0	0
9	0	0	0	0	0	0
10	0	0	0	0	0	0

More moves

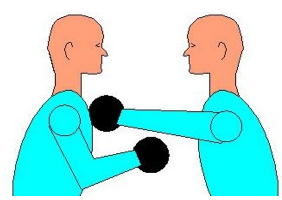
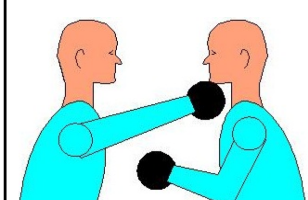
To make the game more interesting, we might consider allowing the fighters to choose among several different types of move. For each possible pair of moves for the two fighters, there may be several possible outcomes with various probabilities; these may involve a fighter losing more than one hit point. I'll suppose there are three possible moves: a punch to the head, a punch to the chest, and a block. Here are the nine possible pairs, the outcomes and their probabilities. The outcomes and probabilities can be changed, but each possible outcome must involve one fighter losing at least one hit point. Of course the probabilities of the outcomes for each pair of moves must add up to 1. In some of the cases there is just one possible outcome, which of course has probability 1.

Notice that if you knew which move your opponent would choose, you would be able to do better: "Block" beats "Head", "Chest" beats "Block" and (on the average) "Head" beats "Chest". Thus this is a game where you are trying to outguess your opponent.

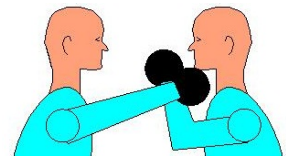
Head vs Head

Probability	You lose	Opponent loses	
.5	3	0	
.5	0	3	

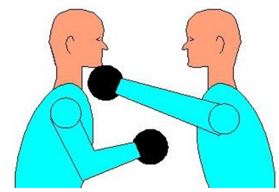
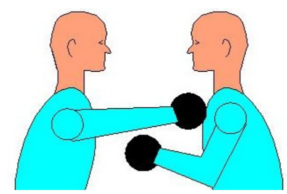
Head vs Chest

.54	2	0	
.46	0	3	

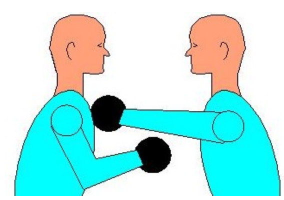
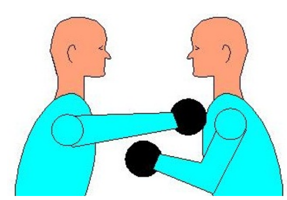
Head vs Block

1.0	1	0	

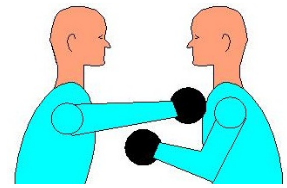
Chest vs Head

.46	3	0	
.54	0	2	

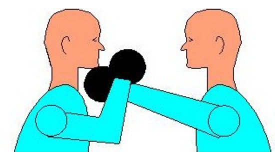
Chest vs Chest

.5	2	0	
.5	0	2	

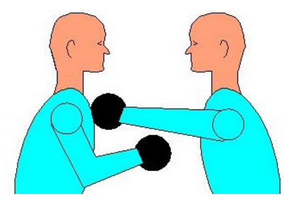
Chest vs Block

1.0	0	2	

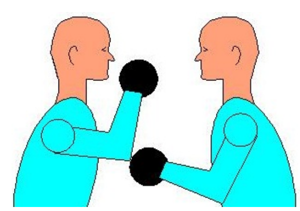
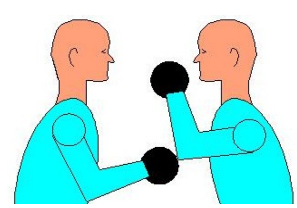
Block vs Head

1.0	0	1	

Block vs Chest

1.0	2	0	

Block vs Block

.5	1	0	
.5	0	1	

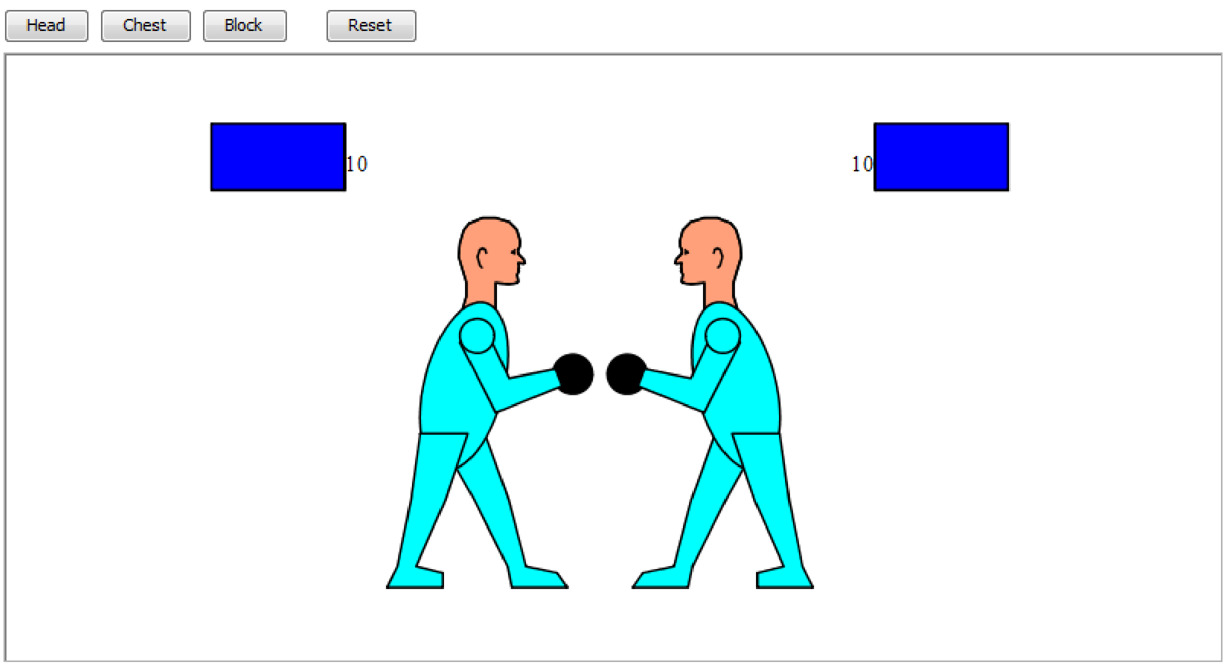
Here is a simulation of the game. The fighter on the right will use a "mixed strategy": at each turn he will randomly choose one of the three possible moves, using probabilities as set below. The probabilities must add up to 1 (and will be adjusted accordingly). You, controlling the fighter on the left, get to choose which move to make by clicking the appropriate button. If you check the "Optimize" box, the opponent's mixed strategy will be optimized: see the section Optimal Play below.

Opponent's strategy: probabilities for

Head: 0.6270005400 Chest: 0.2962508374 Block: 0.0767486226

☑ Optimize

Initial hit points: You 10 Opponent 10



Rounds won: You 0 Opponent 0

Winning probabilities

Let's suppose we know the mixed strategies the two fighters will use at each turn. Calculating the winning probabilities for this game is more complicated than for the first game. We won't get a single formula to do it. Instead, we'll calculate each $W(h_1, h_2)$ in terms of the other $W(i, j)$ where $i \le h_1$ and $j \le h_2$ (and $(i, j) \ne (h_1, h_2)$). We know of course that $W(0, j) = 0$ if $j > 0$ (you've already lost if you have 0 hit points) and $W(i, 0) = 1$ (you've already won if your opponent has 0). We can calculate $W(1, 1)$, and then $W(1, 2)$ using that information, then $W(1, 3)$, and all other $W(1, j)$ that we're interested in; then $W(2, 1)$, $W(2, 2)$, etc.

To calculate $W(h_1, h_2)$, we look at the possibilities for what happens on the first turn. The first fighter chooses his move ("head", "chest" or "block") with probabilities p_1, p_2, p_3 respectively, according to his mixed strategy. The second player independently chooses his move with probabilities q_1, q_2, q_3 according to his mixed strategy. Thus the probability that they choose moves i and j respectively is $p_i q_j$. The outcome may be determined (e.g. for "head" vs "block" fighter 1 loses 1 hit point), or there may be further probabilities to consider: let's say with probability $r(d_1, d_2)$ the outcome is that fighter 1 loses d_1 hit points and fighter 2 loses d_2 (where one of these is 0 and the other positive). This leaves fighter 1 with $h_1 - d_1$ and fighter 2 with $h_2 - d_2$ (with negative values being replaced by 0). We would then be in a situation where your probability of winning is $W(h_1 - d_1, h_2 - d_2)$. To get $W(h_1, h_2)$ we add up all the contributions $p_i q_j r(d_1, d_2)\ W(h_1 - d_1, h_2 - d_2)$ for all i, j, d_1, d_2. For example, with $h_1 = 3$ and $h_2 = 3$ and moves "head" vs "chest", the probabilities are $r(2, 0) = 0.54$, $r(0, 3) = 0.46$, corresponding to terms $0.54\ p_1 q_2 W(1, 3) + 0.46\, p_1 q_2$. It would be rather tedious to do all this by hand, but of course for the computer it's an easy calculation.

Here, then, are your probabilities of winning. For now we'll suppose both players use the same mixed strategy at every turn, as specified below.

Your mixed strategy: probabilities for

Head: 1/3 Chest: 1/3 Block: 1/3

Opponent's mixed strategy: probabilities for

Head: .45 Chest: 0.45 Block: .10

Opponent's Hit Poi...

Your hit points	1	2	3	4	5	6
1	`0.5000`	`0.4167`	`0.3081`	`0.2022`	`0.1520`	`0.1080`
2	`0.5250`	`0.4417`	`0.3324`	`0.2247`	`0.1716`	`0.1244`
3	`0.6792`	`0.5961`	`0.4829`	`0.3643`	`0.2931`	`0.2266`
4	`0.7666`	`0.6861`	`0.5737`	`0.4520`	`0.3718`	`0.2951`
5	`0.8218`	`0.7506`	`0.6490`	`0.5351`	`0.4531`	`0.3723`
6	`0.8735`	`0.8122`	`0.7223`	`0.6175`	`0.5353`	`0.4517`

▼ Optimal play

In the last section we assumed that the same mixed strategies were used no matter how many hit points each fighter had. That's not very realistic. A good strategy when you're winning might not be a good strategy when you're losing. But how can you find the best strategy? What does "best" mean in this context?

Game theory can answer this question. This game is an example of a "two-person zero-sum game", where your payoff is 1 if you win the round and 0 if you lose. Game theory says that each player has an optimal strategy, and the game has a value v: if you use your optimal strategy, you ensure yourself probability at least v of winning the round, no matter what strategy your opponent uses. On the other hand, by using his optimal strategy, your opponent ensures that your probability of winning the round is at most v, no matter what you do. If you use a strategy that is not optimal, you can't do any better than probability v against an opponent who uses his optimal strategy, and you might do worse.

The complete round, involving many different moves, would be very complicated to analyze using game theory. Fortunately, however, each move can be analyzed as a two-person zero-sum game, in which your payoff is your probability of winning the round. Again we will look at each situation where the hit points are (h_1, h_2) after already analyzing all other pairs of hit points (h'_1, h'_2) with $h'_1 \le h_1$ and $h'_2 \le h_2$, so we will know your probability of winning the round given any possible result of this move. Let $w(i, j)$ be the expected probability of winning the round if you choose move i and your opponent chooses move j. To get your optimal mixed strategy, you solve the following linear programming problem:

maximize v
subject to

$$
\begin{aligned}
&v \le w(1,1)\; x_1 + w(2,1)\, x_2 + w(3,1)\, x_3\\
&v \le w(1,2)\, x_1 + w(2,2)\, x_2 + w(3,2)\, x_3\\
&v \le w(1,3)\, x_1 + w(2,3)\, x_2 + w(3,3)\, x_3\\
&x_1 + x_2 + x_3 = 1\\
&x_1, x_2, x_3 \ge 0
\end{aligned}
$$

The interpretation of this is that x_1, x_2, x_3 are your mixed strategies, and v is the value of the game, your probability of winning in this situation with hit points (h_1, h_2). The right side of the first inequality, $w(1,1)\; x_1 + w(2,1)\, x_2 + w(3,1)\, x_3$, would be your probability of winning if you used mixed strategy x_1, x_2, x_3 and the opponent uses move 1 ("head"), and similarly the other two inequalities come from your opponents's other two moves.

For example, with $(h_1, h_2) = (1, 1)$, the first three inequalities are

$$
\begin{aligned}
&v \le 0.5\, x_1 + 0.54\, x_2 + x_3\\
&v \le 0.46\, x_1 + 0.5\, x_2\\
&v \le \qquad\quad\;\; x_2 + 0.5\, x_3
\end{aligned}
$$

where, for example, the $0.54\, x_2$ comes from the fact that if you use "chest" and your opponent uses "head", with probability 0.54 the opponent loses 2 hit points (and you win the round) while with probability 0.46 you lose 3 hit points (and you lose the round). The optimal solution for this problem

is $x_1 = 0$, $x_2 = 1$, $x_3 = 0$, $v = 0.5$. Thus linear programming is saying that in this situation you should always use "chest", obtaining a probability of winning at least 0.5. The opponent's optimal strategy is also to always use "chest". If your opponent chooses anything else when you use "chest", it would increase your probability of winning. If you choose anything else when your opponent uses "chest", it would decrease your probability of winning.
It's easy to see why this is the case. With both fighters having just 1 hit point, "chest" is a better move than "head", no matter what your opponent does. And once you know that your opponent is not going to use "head", there's no reason to use "block" either.

On the other hand, with $(h_1, h_2) = (3, 3)$, if you use "chest" and your opponent uses "head" and the opponent loses 2 hit points, the hit points will go to $(3, 1)$, which is not a sure win. From the linear programming problem for hit points $(3, 1)$, your probability of winning in that situation is 0.6760563380. So instead of .54, the coefficient of x_2 in the first inequality would be $0.54 \cdot 0.6760563380 = 0.3650704225$. The three inequalities for this linear programming problem turn out to be

$$v \le 0.5\, x_1 + 0.3650704225\, x_2 + 0.6140094856\, x_3,$$
$$v \le 0.6349295775\, x_1 + 0.5\, x_2 + 0.3239436620\, x_3,$$
$$v \le 0.3859905144\, x_1 + 0.6760563380\, x_2 + 0.5\, x_3$$

Note that in this case there would be an advantage to using "head" if you knew your opponent was using "chest", because $0.6349295775 > 0.5$. None of the moves is better than another in all situations, and in fact the optimal solution here involves all three moves:
$x_1 = 0.4142546897$, $x_2 = 0.2682605160$, $x_3 = 0.3174847943$, $v = 0.5$.

Your opponent's optimal mixed strategies can be obtained from a similar linear programming problem, which turns out to be the dual of the one for your optimal mixed strategies.

The following table presents your probability of winning with optimal play, for each pair (h_1, h_2) :

Opponent's Hit Poi...

Your hit points	1	2	3	4	5	6
1	\`0.5000\`	\`0.5000\`	\`0.3239\`	\`0.2500\`	\`0.1901\`	\`0.1360\`
2	\`0.5000\`	\`0.5000\`	\`0.3860\`	\`0.2500\`	\`0.2237\`	\`0.1644\`
3	\`0.6761\`	\`0.6140\`	\`0.5000\`	\`0.3842\`	\`0.3192\`	\`0.2520\`
4	\`0.7500\`	\`0.7500\`	\`0.6158\`	\`0.5000\`	\`0.4461\`	\`0.3554\`
5	\`0.8099\`	\`0.7763\`	\`0.6808\`	\`0.5539\`	\`0.5000\`	\`0.4171\`
6	\`0.8640\`	\`0.8356\`	\`0.7480\`	\`0.6446\`	\`0.5829\`	\`0.5000\`

Here you can see the optimal strategies for you and your opponent for any given (h_1, h_2) :

Your hit points: 2 Opponent's hit points: 6

Your optimal mixed strategy:

Head	0.7502961993	Chest	0.2497038007	Block	0.0000000000

Opponent's optimal mixed strategy:

Head	0.0000000000	Chest	0.6845416422	Block	0.3154583578

If your opponent is using an optimal mixed strategy, your expected payoff will be the same for all of your moves that have nonzero probabilities in your optimal mixed strategy. So e.g. if you have 2 hit points and your opponent has 6, your optimal mixed strategy involves "Head" and "Chest" but not "Block". If you know the opponent is using his optimal mixed strategy, you could actually use any combination of "Head" and "Chest" and have the same expected payoff: in this case probability 0.1644 of winning. However, against a clever opponent it would be dangerous to use anything other than your optimal mixed strategy, because if the opponent could figure out which strategy you were using he could adjust his strategy to take advantage of that.

My son points out that in real fighting games there is an element of skill, and we don't have set probabilities for the various outcomes. Presumably, for a particular player who has attained a certain level of skill, these probabilities exist, we just don't know them. If you played the game many times you could gather enough statistics on how many times the various outcomes occurred in order to estimate the probabilities. This is left as an exercise...