Classroom Tips and Techniques: Simultaneous Diagonalization and the Generalized Eigenvalue Problem

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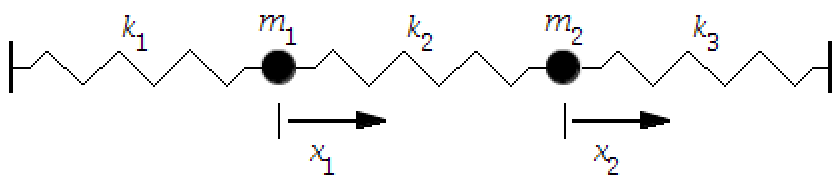


Introduction

This article explores the connections between the generalized eigenvalue problem and the problem of simultaneously diagonalizing a pair of $n\times n$ matrices.

Given the $n\times n$ matrices A and B, the *generalized eigenvalue problem* seeks the eigenpairs $(\lambda_k, \mathbf{x}_k)$, solutions of the equation $A\mathbf{x} = \lambda B\mathbf{x}$, or $(A - \lambda B)\mathbf{x} = \mathbf{0}$. If B is nonsingular, the eigenpairs of $B^{-1}A$ are solutions. If a matrix S exists for which $S^T A S = \Lambda$, and $S^T B S = I$, where Λ is a diagonal matrix and I is the $n\times n$ identity, then A and B are said to be *diagonalized simultaneously*, in which case the diagonal entries of Λ are the generalized eigenvalues for A and B. Such a matrix S exists if A is symmetric and B is positive definite. (Our definition of positive definite includes symmetry.)

The **Eigenvector** command in the *LinearAlgebra* package will return the generalized eigenpairs, which can also be obtained by a variety of brute-force calculations. There is single command in Maple for finding S, the transition matrix for the simultaneous diagonalization problem. Note, finally, that in general, S is not an orthogonal matrix, so the transformations of A and B to diagonal form are not similarity transformations.

Motivation

Provided friction is ignored, the spring-mass system sketched in Figure 1 is modeled by the differential equations

$$m_1 x_1'' + (k_1 + k_3)\, x_1 - k_3 x_2 = 0$$
$$m_2 x_2'' - k_3 x_1 + (k_2 + k_3)\, x_2 = 0$$

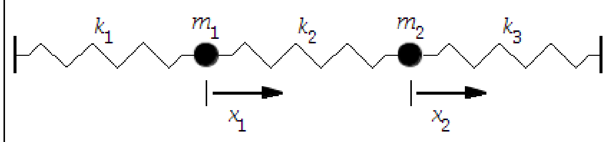


Figure 1 Spring-mass system

These equations can be cast into the form $M\,\mathbf{x}'' + K\,\mathbf{x} = \mathbf{0}$, provided the following definitions are made.

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \qquad K = \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Simultaneous diagonalization of the matrices K and M means that $S^{\mathrm{T}} M S = I$ and $S^{\mathrm{T}} K S = \Lambda$, so that $M = (S^{\mathrm{T}})^{-1} S^{-1}$ and $K = (S^{\mathrm{T}})^{-1} \Lambda S^{-1}$. Hence, the differential equation becomes

$$(S^{\mathrm{T}})^{-1} S^{-1} \mathbf{x}'' + (S^{\mathrm{T}})^{-1} \Lambda S^{-1} \mathbf{x} = \mathbf{0}$$

or

$$S^{-1} \mathbf{x}'' + \Lambda S^{-1} \mathbf{x} = \mathbf{0}$$

upon multiplying from the left by S^{T} . Defining $\mathbf{X} = S^{-1}\mathbf{x}$, or equivalently, $\mathbf{x} = S\mathbf{X}$, puts the differential equation into the separated form $\mathbf{X}'' + \Lambda\,\mathbf{X} = \mathbf{0}$, from which a solution is immediately available. The square roots of the diagonal elements in Λ are the angular frequencies for the normal modes of oscillation of this undamped vibrating system.

Alternatively, if solutions of the form $\mathbf{x} = e^{\lambda t}\mathbf{v}$ are sought, where **v** is a constant vector, the differential equation becomes $\lambda^2 M\,\mathbf{v} + K\,\mathbf{v} = \mathbf{0}$, a generalized eigenvalue problem.

Example 1

If $A = \begin{bmatrix} 7 & 1 \\ 1 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 2 \\ 2 & 7 \end{bmatrix}$, the eigenpairs for the generalized eigenvalue problem are

$$\left(43/10, \begin{bmatrix} -19/4 \\ 1 \end{bmatrix}\right) \text{ and } \left(-1, \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}\right)$$

A matrix S that simultaneously diagonalizes A and B is

$$\begin{bmatrix} -\frac{19}{530}\sqrt{106}\sqrt{5} & -\frac{1}{265}\sqrt{265}\sqrt{5} \\ \frac{2}{265}\sqrt{106}\sqrt{5} & \frac{3}{265}\sqrt{265}\sqrt{5} \end{bmatrix}$$

That A and B are symmetric is obvious by inspection; that B is positive definite can be established by showing its eigenvalues are positive or by applying the **IsDefinite** command from *LinearAlgebra*.

Initialize Maple and enter the data:

• Tools: Load Package≻Student Linear Algebra	Loading Student:-LinearAlgebra
• Control-drag the equation $A = \ldots$ • Context Menu: Assign Name	$A = \begin{bmatrix} 7 & 1 \\ 1 & -6 \end{bmatrix} \xrightarrow{\text{assign}}$
• Control-drag the equation $B = \ldots$ • Context Menu: Assign Name	$B = \begin{bmatrix} 2 & 2 \\ 2 & 7 \end{bmatrix} \xrightarrow{\text{assign}}$

Show that B is positive definite:

• Type B and press the Enter key. • Context Menu: Eigenvalues, etc≻Eigenvalues • Context Menu: Approximate≻5	B $\begin{bmatrix} 2 & 2 \\ 2 & 7 \end{bmatrix}$ $\xrightarrow{\text{eigenvalues}}$ $\begin{bmatrix} \frac{9}{2} + \frac{1}{2}\sqrt{41} \\ \frac{9}{2} - \frac{1}{2}\sqrt{41} \end{bmatrix}$ $\xrightarrow{\text{at 5 digits}}$ $\begin{bmatrix} 7.7016 \\ 1.2984 \end{bmatrix}$
• Apply the **IsDefinite** command.	$IsDefinite(B, query = positive\_definite) = true$

To obtain the eigenpairs that satisfy the generalized eigenvalue problem, use the **Eigenvectors** command:

$$Eigenvectors(A, B) \;=\; \begin{bmatrix} -1 \\ \frac{43}{10} \end{bmatrix}, \begin{bmatrix} -\frac{1}{3} & -\frac{19}{4} \\ 1 & 1 \end{bmatrix}$$

The vector on the right is a vector of eigenvalues. The columns in the matrix on the right are the corresponding eigenvectors. The order in which the eigenpairs are returned is not fixed. Repeated execution of the same command can reverse the order. The matrix S given above is based on the order $\lambda_1 = 43/10$, $\lambda_2 = -1$.

The Generalized Eigenvalue Problem

The **Eigenvectors** command has provided the generalized eigenpairs for the matrices A and B in Example 1. These eigenpairs are also available as the eigenpairs of the matrix $F = B^{-1}A$, as can be seen from

$$Eigenvectors\left(B^{-1}.A\right) \;=\; \begin{bmatrix} \frac{43}{10} \\ -1 \end{bmatrix}, \begin{bmatrix} -\frac{19}{4} & -\frac{1}{3} \\ 1 & 1 \end{bmatrix}$$

Alternatively, the generalized eigenpairs can also be obtained from first principles with the following calculations.

Define the matrix $C = A - \lambda B$	
• Expression palette: $f := a \rightarrow y$ • $C := \lambda \rightarrow A - \lambda B$	$C := \lambda \rightarrow A - \lambda B$:
Obtain and solve $\lvert C(\lambda)\rvert = 0$	
• Type $C(\lambda)$ and press the Enter key. • Context Menu: Standard Operations>Determinant • Context Menu: Solve>Solve	$C(\lambda)$ $\begin{bmatrix} 7 - 2\lambda & 1 - 2\lambda \\ 1 - 2\lambda & -6 - 7\lambda \end{bmatrix}$ $\xrightarrow{\text{determinant}}$ $-43 - 33\lambda + 10\lambda^2$ $\xrightarrow{\text{solve}}$ $\left\{\lambda = \frac{43}{10}\right\}, \{\lambda = -1\}$

Obtain the generalized eigenvector corresponding to $\lambda = -1$:

• Type $C(-1)$ and press the Enter key. • Context Menu: Vector Spaces≻Null Space	$C(-1)$ $\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$ $\xrightarrow{\text{null space}}$ $\left[\left\{\left[\begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}\right]\right\}\right]$
Eigenvector from first principles	
• Enter $C(-1)\mathbf{x}$, where $\mathbf{x} = \langle a, b\rangle$ • Context Menu: Conversions≻To List • Context Menu: Solve≻Solve for Variables≻a • Expression palette: Evaluation template Evaluate **x** at the solution for a • Context Menu: Evaluate at a Point≻$b = 1$	$C(-1).\langle a, b\rangle$ $\begin{bmatrix} 9a + 3b \\ 3a + b \end{bmatrix}$ $\xrightarrow{\text{to list}}$ $[9a + 3b, 3a + b]$ $\xrightarrow{\text{solve (specified)}}$ $\left\{a = -\frac{1}{3}b\right\}$ $\langle a, b\rangle\Big\vert_{a=-\frac{b}{3}}$ $\begin{bmatrix} -\frac{1}{3}b \\ b \end{bmatrix}$ $\xrightarrow{\text{evaluate at point}}$ $\begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$

Obtain the generalized eigenvector corresponding to $\lambda = 43/10$:

• Type $C(43/10)$ and press the Enter key. • Context Menu: Vector Spaces≻Null Space	$C(43/10)$ $\begin{bmatrix} -\frac{8}{5} & -\frac{38}{5} \\ -\frac{38}{5} & -\frac{361}{10} \end{bmatrix}$ $\xrightarrow{\text{null space}}$ $\left[\left\{\left[\begin{bmatrix} -\frac{19}{4} \\ 1 \end{bmatrix}\right]\right\}\right]$

Eigenvector from first principles	
• Enter $C(43/10)\mathbf{x}$, where $\mathbf{x} = \langle a, b\rangle$	$C(43/10).\langle a, b\rangle$ $\begin{bmatrix} -\frac{8}{5}a - \frac{38}{5}b \\ -\frac{38}{5}a - \frac{361}{10}b \end{bmatrix}$
• Context Menu: Conversions≻To List	$\xrightarrow{\text{to list}}$ $\left[-\frac{8}{5}a - \frac{38}{5}b, -\frac{38}{5}a - \frac{361}{10}b\right]$
• Context Menu: Solve≻Solve for Variables≻a	$\xrightarrow{\text{solve (specified)}}$ $\left\{a = -\frac{19}{4}b\right\}$
• Expression palette: Evaluation template Evaluate **x** at the solution for a	$\langle a, b\rangle \Big\vert_{a=-\frac{19}{4}b}$ $\begin{bmatrix} -\frac{19}{4}b \\ b \end{bmatrix}$
• Context Menu: Evaluate at a Point≻$b = 1$	$\xrightarrow{\text{evaluate at point}}$ $\begin{bmatrix} -\frac{19}{4} \\ 1 \end{bmatrix}$

Simultaneous Diagonalization

The following prescription for finding a matrix S that simultaneously diagonalizes A and B is taken from [1, p.516], [2, p.1151], and [3, p.429].

Let $(\lambda_k, \mathbf{v}_k)$ be the eigenpairs of the matrix $L^{-1}A(L^{-1})^{\mathrm{T}}$, where the $\mathbf{v}_k$ are unit vectors, and L is the lower triangular factor in the Cholesky decomposition $B = L\,L^{\mathrm{T}}$; then $S = (L^{-1})^{\mathrm{T}}[\mathbf{v}_1, \ldots, \mathbf{v}_n]$.

This algorithm can be implemented through the Context Menu, as demonstrated below. Unfortunately, the options available when the Student *LinearAlgebra* package is installed are not the same as those available with the *LinearAlgebra* package itself.

Initialize Maple:

• Tools≻Unload Package≻Student Linear Algebra	Unloading Student:-LinearAlgebra
• Tools≻Load Package≻Linear Algebra	Loading LinearAlgebra

Obtain the Cholesky decomposition of B and the factor L :

• Enter the matrix B • Context Menu: Solvers and Forms≻Cholesky • Context Menu: Assign to a Name≻L	B $\begin{bmatrix} 2 & 2 \\ 2 & 7 \end{bmatrix}$ $\xrightarrow{\text{Cholesky decomposition}}$ $\begin{bmatrix} \sqrt{2} & 0 \\ \sqrt{2} & \sqrt{5} \end{bmatrix}$ $\xrightarrow{\text{assign to a name}}$ L

Obtain the matrix $M = [\mathbf{v}_1, \mathbf{v}_2]$:

• Compute $L^{-1}A(L^{-1})^{T}$. • Context Menu: Eigenvalues, etc≻Eigenvectors • Context Menu: Select Element≻2 • Context Menu: Select Elements≻Split into Columns	$L^{-1}.A.(L^{-1})^{\%T}$ $\begin{bmatrix} \frac{7}{2} & -\frac{3}{5}\sqrt{2}\sqrt{5} \\ -\frac{3}{5}\sqrt{2}\sqrt{5} & -\frac{1}{5} \end{bmatrix}$ $\xrightarrow{\text{eigenvectors}}$ $\begin{bmatrix} \frac{43}{10} \\ -1 \end{bmatrix},$ $\begin{bmatrix} -\frac{3}{4}\sqrt{2}\sqrt{5} & \frac{2}{15}\sqrt{2}\sqrt{5} \\ 1 & 1 \end{bmatrix}$ $\xrightarrow{\text{select entry 2}}$ $\begin{bmatrix} -\frac{3}{4}\sqrt{2}\sqrt{5} & \frac{2}{15}\sqrt{2}\sqrt{5} \\ 1 & 1 \end{bmatrix}$ $\xrightarrow{\text{split into columns}}$

	$\left[\begin{bmatrix} -\frac{3}{4}\sqrt{2}\sqrt{5} \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{2}{15}\sqrt{2}\sqrt{5} \\ 1 \end{bmatrix}\right]$
• Context Menu: • Normalized (mapped)≻Eulcidean	$\xrightarrow{\text{Euclidean-normalize}}$ $\left[\begin{bmatrix} -\frac{3}{106}\sqrt{106}\sqrt{2}\sqrt{5} \\ \frac{2}{53}\sqrt{106} \end{bmatrix}, \begin{bmatrix} \frac{2}{265}\sqrt{265}\sqrt{2}\sqrt{5} \\ \frac{3}{53}\sqrt{265} \end{bmatrix}\right]$
• • • • •	
• Context Menu: Select Elements≻Combine into Matrix	$\xrightarrow{\text{combine into Matrix}}$ $\left[\left[-\frac{3}{106}\sqrt{106}\sqrt{2}\sqrt{5}, \frac{2}{265}\sqrt{265}\sqrt{2}\sqrt{5}\right], \left[\frac{2}{53}\sqrt{106}, \frac{3}{53}\sqrt{265}\right]\right]$
• Context Menu: Assign to a Name≻M	$\xrightarrow{\text{assign to a name}}$ M

Obtain the matrix S:

• Obtain the matrix $\left(L^{-1}\right)^{\mathrm{T}}\left[\mathbf{v}_1, \mathbf{v}_2\right]$	$\left(L^{-1}\right)^{\%T}.M$ $\left[\left[-\frac{19}{530}\sqrt{106}\sqrt{5}, -\frac{1}{265}\sqrt{265}\sqrt{5}\right], \left[\frac{2}{265}\sqrt{106}\sqrt{5}, \frac{3}{265}\sqrt{265}\sqrt{5}\right]\right]$
• Context Menu: Assign to a Name≻S	$\xrightarrow{\text{assign to a name}}$ S

Verify the simultaneous diagonalizations $S^{\mathrm{T}} A\, S = \Lambda$ and $S^{\mathrm{T}} B\, S = I$:

• Enter $S^{T} A\, S$ Context Menu: Evaluate and Display Inline	$S^{\%T}.A.S \;=\; \begin{bmatrix} \frac{43}{10} & 0 \\ 0 & -1 \end{bmatrix}$
• Enter $S^{T} B\, S$ Context Menu: Evaluate and Display Inline	$S^{\%T}.B.S \;=\; \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

The order in which Maple returns the eigenpairs of the matrix $L^{-1} A \left(L^{-1}\right)^{T}$ determines which of the four possible forms for S these computations will produce. That there are four such possibilities can be established from first principles, as shown in the following section.

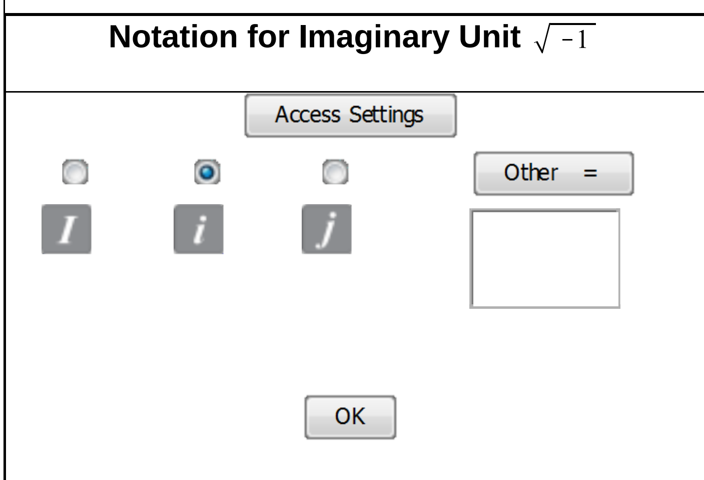
Simultaneous Diagonalization from First Principles

Define I as the 2 × 2 identity matrix. In order to use the symbol I for the identity matrix, its connection to the imaginary unit $\sqrt{-1}$ must be broken. This can be done with the command

$$interface(imaginaryunit = i)$$

or with the following task template.

Tools>Tasks>Browse:
Algebra>Complex Arithmetic>Set Imaginary Unit



Define matrices S, Λ, and I:

• Matrix palette: 2 × 2 matrix template • Context Menu: Assign to a Name≻S	$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{\text{assign to a name}} S$
Matrix palette: 2 × 2 matrix template • Context Menu: Assign to a Name≻Lambda	$\begin{bmatrix} 43/10 & 0 \\ 0 & -1 \end{bmatrix} \xrightarrow{\text{assign to a name}} \Lambda$
Matrix palette: × matrix • Context Menu: Assign to a Name≻I	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{\text{assign to a name}} I$

Equate corresponding components in $S^T A\, S$ and Λ:

• Form $S^T A\, S$	$S^{\%T}.A.S, \Lambda$ $[[(7a+c)\,a+(a-6c)\,c, (7a+c)\,b+(a-6c)\,d], [(7b+d)\,a+(b-6d)\,c, (7b+d)\,b+(b-6d)\,d]], \begin{bmatrix} \frac{43}{10} & 0 \\ 0 & -1 \end{bmatrix}$
• Context Menu: Equate	$\xrightarrow{\text{equate}}$ $\left[(7a+c)\,a+(a-6c)\,c=\frac{43}{10}, (7a+c)\,b+(a-6c)\,d=0, (7b+d)\,a+(b-6d)\,c=0, (7b+d)\,b+(b-6d)\,d=-1\right]$
• Context Menu: Conversions≻To Set	$\xrightarrow{\text{to set}}$ $\left\{(7a+c)\,a+(a-6c)\,c=\frac{43}{10}, (7a+c)\,b+(a-6c)\,d=0, (7b+d)\,a+(b-6d)\,c=0, (7b+d)\,b+(b-6d)\,d=-1\right\}$

Equate corresponding components in $S^T B\, S$ and I:

• Form $S^T B\, S$	$S^{\%T}.B.S, I$

• Context Menu: Equate • Context Menu: Conversions≻To Set	$\left[\left[(2a+2c)a+(2a+7c)c,(2a+2c)b+(2a+7c)d\right],\left[(2b+2d)a+(2b+7d)c,(2b+2d)b+(2b+7d)d\right]\right],\begin{bmatrix}1&0\\0&1\end{bmatrix}$ $\xrightarrow{\text{equate}}$ $[(2a+2c)a+(2a+7c)c=1,(2a+2c)b+(2a+7c)d=0,(2b+2d)a+(2b+7d)c=0,(2b+2d)b+(2b+7d)d=1]$ $\xrightarrow{\text{to set}}$ $\{(2a+2c)a+(2a+7c)c=1,(2a+2c)b+(2a+7c)d=0,(2b+2d)a+(2b+7d)c=0,(2b+2d)b+(2b+7d)d=1\}$

Join the two sets of equations and solve the eight equations for the four unknowns a, b, c, d :

• Reference the sets by their equation labels and form a set union with the union operator selected from the Common Symbols palette. Context Menu: Solve≻Solve (explicit)	**(1)**∪ **(2)** $\left\{(2a+2c)a+(2a+7c)c=1,(2a+2c)b+(2a+7c)d=0,(7a+c)a+(a-6c)c=\frac{43}{10},(7a+c)b+(a-6c)d=0,(2b+2d)a+(2b+7d)c=0,(2b+2d)b+(2b+7d)d=1,(7b+d)a+(b-6d)c=0,(7b+d)b+(b-6d)d=-1\right\}$ $\xrightarrow{\text{solve}}$

	$\left\{a=\frac{19}{530}\sqrt{530}, b=\frac{1}{53}\sqrt{53}, c=-\frac{2}{265}\sqrt{530}, d=-\frac{3}{53}\sqrt{53}\right\}, \left\{a=-\frac{19}{530}\sqrt{530}, b=\frac{1}{53}\sqrt{53}, c=\frac{2}{265}\sqrt{530}, d=-\frac{3}{53}\sqrt{53}\right\}, \left\{a=\frac{19}{530}\sqrt{530}, b=-\frac{1}{53}\sqrt{53}, c=-\frac{2}{265}\sqrt{530}, d=\frac{3}{53}\sqrt{53}\right\}, \left\{a=-\frac{19}{530}\sqrt{530}, b=-\frac{1}{53}\sqrt{53}, c=\frac{2}{265}\sqrt{530}, d=\frac{3}{53}\sqrt{53}\right\}$

Obtain the four possible solutions for the matrix S:

• Expression palette: Evaluation template Reference the first solution. • Context Menu: Evaluate and Display Inline	$S\Big\|_{(3)[1]} = \begin{bmatrix} \frac{19}{530}\sqrt{530} & \frac{1}{53}\sqrt{53} \\ -\frac{2}{265}\sqrt{530} & -\frac{3}{53}\sqrt{53} \end{bmatrix}$
• Expression palette: Evaluation template Reference the second solution. • Context Menu: Evaluate and Display Inline	$S\Big\|_{(3)[2]} = \begin{bmatrix} -\frac{19}{530}\sqrt{530} & \frac{1}{53}\sqrt{53} \\ \frac{2}{265}\sqrt{530} & -\frac{3}{53}\sqrt{53} \end{bmatrix}$
• Expression palette: Evaluation template Reference the third solution. • Context Menu: Evaluate and Display Inline	$S\Big\|_{(3)[3]} = \begin{bmatrix} \frac{19}{530}\sqrt{530} & -\frac{1}{53}\sqrt{53} \\ -\frac{2}{265}\sqrt{530} & \frac{3}{53}\sqrt{53} \end{bmatrix}$
• Expression palette: Evaluation template Reference the fourth solution. • Context Menu: Evaluate and Display Inline	$S\Big\|_{(3)[4]} = \begin{bmatrix} -\frac{19}{530}\sqrt{530} & -\frac{1}{53}\sqrt{53} \\ \frac{2}{265}\sqrt{530} & \frac{3}{53}\sqrt{53} \end{bmatrix}$

Careful inspection of these four solutions shows that two effects are at play. The order of the eigenvectors is the first influence, and the orientation of the unit eigenvectors is the other. Since each agent can be in two states, there are then four possible solutions for S.

References

- Numerical Analysis: A Practical Approach, 3rd ed., Melvin J. Marin and Robert J. Lopez, Wadsworth Publishing Company, 1991.

- Advanced Engineering Mathematics, Robert J. Lopez, Addison-Wesley Publishing Company, 2001. Now available from Maplesoft as the ebook Advanced Engineering Mathematics with Maple.

- Applied Linear Algebra, 3rd ed., Ben Noble and James W. Daniel, Prentice-Hall Publishing Company, 1988.