Great Expectations, Almost Sure Failure

An investor, Rosie, is offered what appears to be a great opportunity to invest in Risky Resources. Every week the value of her investment will either increase by 40% (with probability 0.55) or decrease by 40% (with probability 0.45), the results in different weeks being statistically independent, and this will continue indefinitely. (In the real world, Rosie should be skeptical and ask how we know that this is the case. But this is an example for a Mathematics course, where the author of a problem has unlimited power to set up the conditions of that problem, and the student must have complete faith in those conditions)

Having studied Probability, Rosie can calculate the expected return on this investment. If the value of her initial investment is W_0, then after one week the value W_1 will be either $1.4\ W_0$ with probability 0.55 or $0.6\ W_0$ with probability 0.45, so the expected value is

$$E(W_1) = 0.55\ (1.4\ W_0) + 0.45\ (0.6\ W_0) = 1.04\ W_0$$

This will continue in future weeks. I'll define a sequence of independent random variables X_i so that

$$X_i = \begin{cases} 1.4 & \text{with probability } 0.55 \\ 0.6 & \text{with probability } 0.45 \end{cases}$$

and $W_i = X_i\, W_{i-1}$. Thus after n weeks her investment's value is $W_n = X_1 X_2 \ldots X_n\, W_0$. Since each $E(X_i) = 1.04$ and the X_i are independent,

$$E(W_n) = E(X_1)\ E(X_2)\ \ldots\ E(X_n)\ W_0 = 1.04^n\ W_0$$

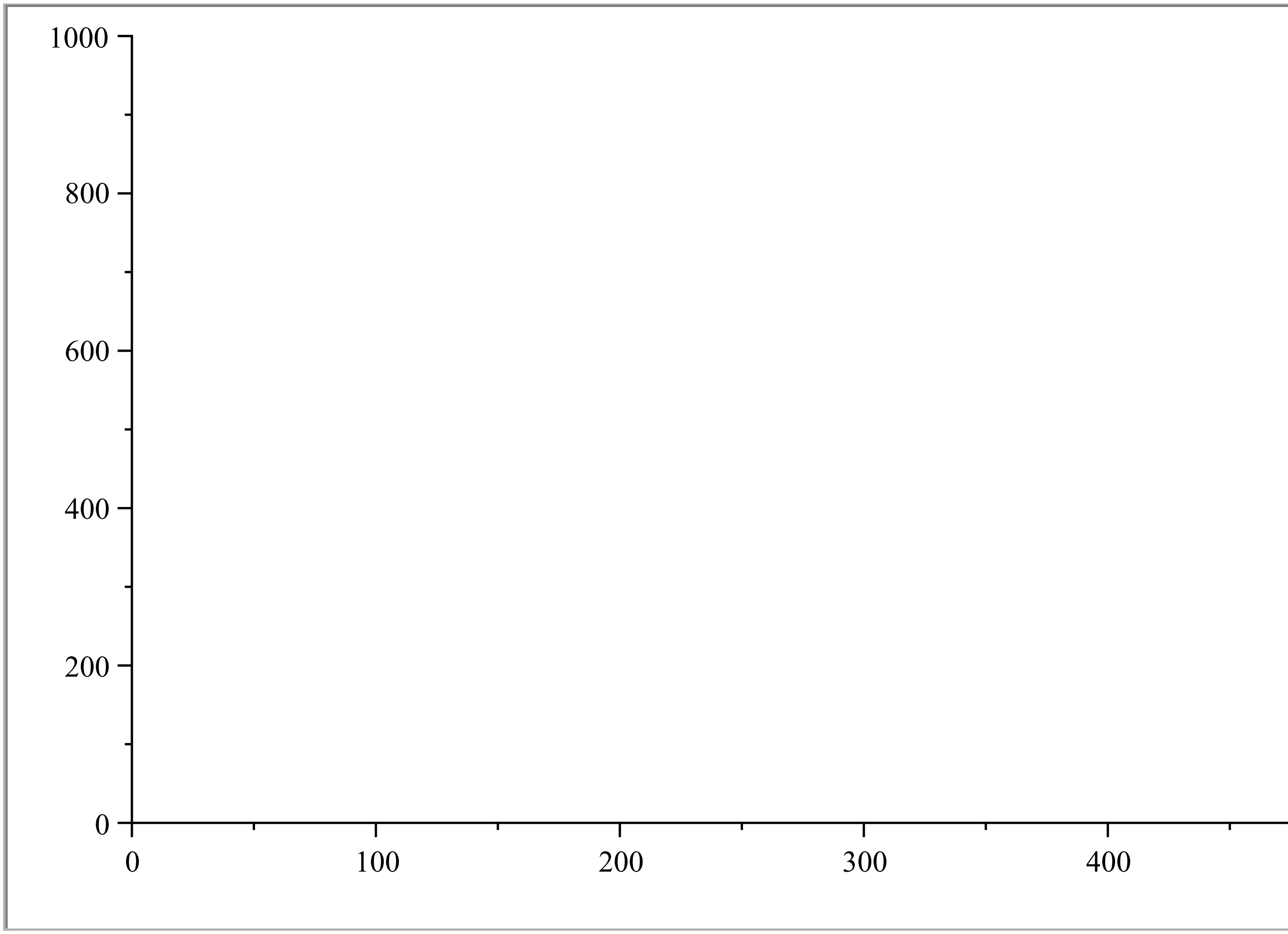
Four percent expected growth per week! In one 52-week year, on the average, her initial investment has been multiplied by 1.04^{52} which is almost 7.7. Move over, Bill Gates and Warren Buffett, here comes Rosie!

Unfortunately, this picture is not borne out by reality.
Pressing the "Go" button below will simulate this process several times, starting with the same initial investment, and plot the results (each simulation in a different colour). The number of simulations and the duration in weeks can be specified. The plot can be viewed with y axis in either linear mode (where each unit of height corresponds to a fixed amount of money) or logarithmic mode (which is good for seeing what happens when the value becomes very large or very small).

Initial investment value	1000
Number of simulations	5
Duration in weeks	500

○ Linear mode ◉ Logarithmic mode

Go



Although Rosie's investment may do very well for a while (sometimes spectacularly so), eventually its value will decline towards 0. What is going on? Was there something wrong with the expected value calculation?

Expected versus Almost Sure

What's actually happening here is that there is a disconnect between the expected value and the almost sure behaviour. In effect, Rosie is betting on enormous returns which have extremely low probabilities. To get a more realistic picture of the situation, we need to see what will happen with high probability, in fact with probability 1, in the limit as time goes to ∞.

Now the usual limit theorems of probability that talk about "probability 1" deal with sums of independent random variables. What we have here is a product rather than a sum. Fortunately, we can convert a product to a sum using the logarithm function. Thus I will define $Y_i = \log(X_i)$ and $S_n = \sum_{i=1}^{n} Y_i$ so that $W_n = W_0 \, e^{S_n}$. We can then apply the Law of Large Numbers and Central Limit

Theorem to S_n. We can start by calculating $E(Y_i)$.

$E(Y_i) = 0.55 \log(1.4) + 0.45 \log(0.6) = -.0448118006$

This is a negative number. According to the Law of Large Numbers, with probability 1 we will have $\frac{S_n}{n} \rightarrow E(Y_i)$ as $n \rightarrow \infty$, and in particular $S_n \rightarrow -\infty$, which means $W_n \rightarrow 0$. That is, it is certain that the investment value will approach 0 as time goes on. Of course it can never get to exactly 0, but eventually it will be close enough for all practical purposes.

More detail can be obtained using the Central Limit Theorem. If Y_i has mean μ and standard deviation σ, for large n we can approximate S_n by a normal distribution with mean $n\mu$ and standard deviation $\sqrt{n}\,\sigma$. As above, $\mu = -.0448118006$, and it's not hard to calculate $\sigma = (\log(1.4) - \log(0.6))\sqrt{0.55\,(1 - 0.55)} = 0.4215253633$.
We find, for example, that S_{2000} has mean $2000\,\mu \approx -89.6$ and standard deviation $\sqrt{2000}\,\sigma \approx 18.9$. The probability that $S_{2000} \geq 0$, i.e. that after 2000 weeks the investment is still worth at least as much as its initial value, is approximately the probability that a normal random variable with this mean and standard deviation is nonnegative. That's extremely small (approximately 10^{-6}). Rosie might as well hope to win a lottery as hope to break even after 2000 weeks. Even without doing the actual calculation you can see that the result should be very small, because 0 is about $\frac{89.6}{18.9} = 4.74$ standard deviations above the mean. Anything more than about 3 standard deviations above the mean for a normal distribution has very low probability.

In the table below, you can enter a number of weeks, an initial investment and an investment target. When you press the Go button, Maple will calculate the probability that after that many weeks Rosie's investment will be worth at least as much as the target. This is done in two ways: "Probability to achieve" is calculated using the Binomial distribution for the number of times the investment gains in the given time period, and "Normal approximation" is calculated using a normal distribution (with continuity correction).

Duration in weeks	500
Initial investment	
Investment target	

Go

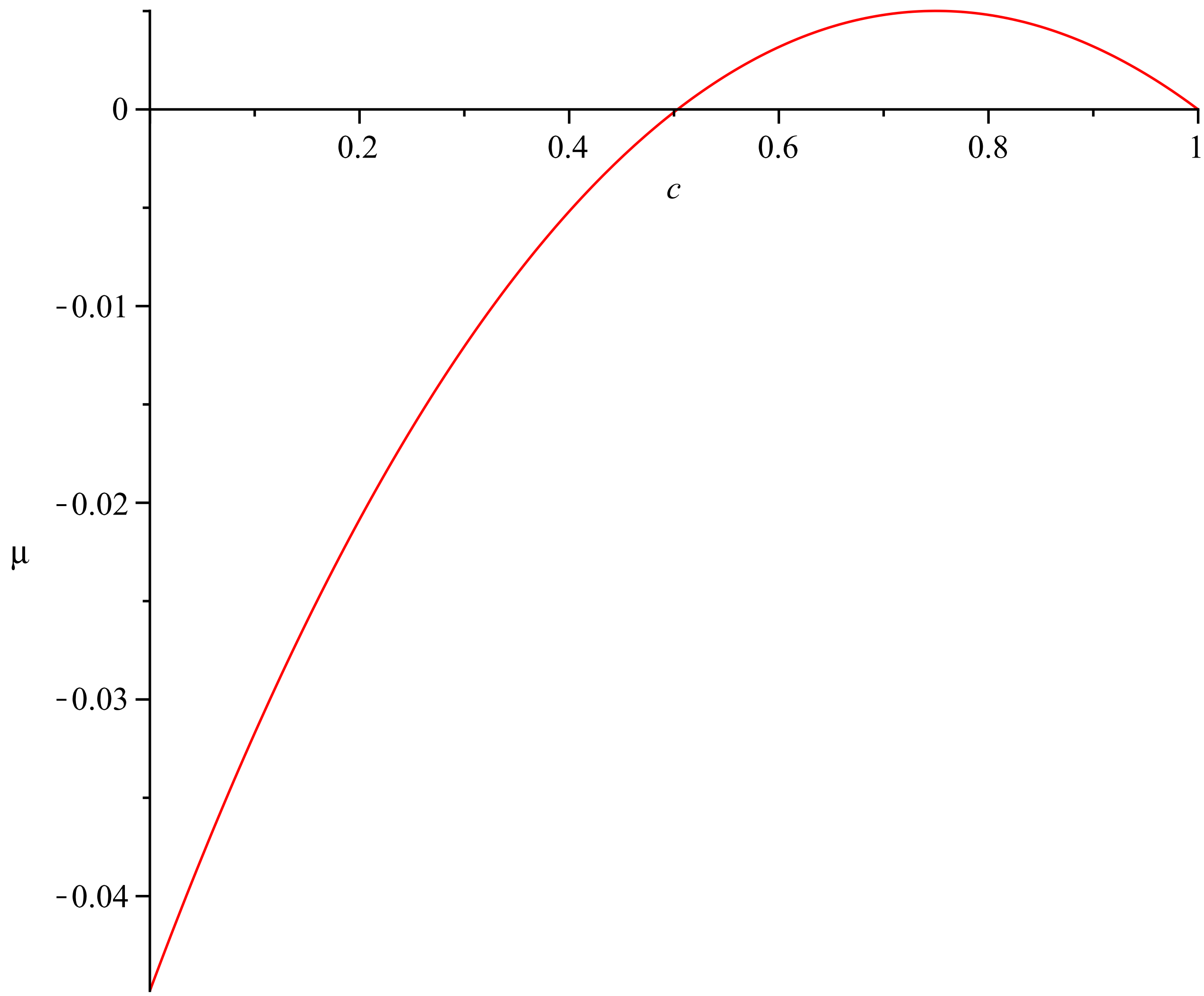
Probability to achieve	
Normal approximation	

What should Rosie do?

There are many strategies that Rosie could use to avoid having the investment's value go almost surely to 0, at the cost of a more modest growth in expected value. One piece of good advice that one always hears is "Diversify! Don't keep all your eggs in one basket". That can work here too.

Let's suppose instead of putting everything into this investment, Rosie prudently keeps a certain fraction c of her wealth in cash and puts the other $1 - c$ into the investment. The cash earns no interest, but it doesn't lose any value either. Each week, as the investment goes up or down, she will either sell some or buy more so that the fraction of cash remains equal to c (I'm ignoring any transaction costs for doing this). In a week where the investment gains 40%, her total wealth is multiplied by $c + (1 - c)\,1.4 = 1.4 - 0.4\,c$, while in a week where the investment loses 40%, it is multiplied by $c + (1 - c)\,0.6 = 0.6 + 0.4\,c$. Thus Rosie's expected value is multiplied by $0.55\,(1.4 - 0.4\,c) + 0.45\,(0.6 + 0.4\,c) = 1.04 - 0.04\,c$.
Of course, this is less than what the expected value would be with $c = 0$, but the benefit will become apparent if we do our calculation with logarithms as before.

The new μ is $0.55\log(1.4 - 0.4\,c) + 0.45\log(0.6 + 0.4\,c)$. As we saw, that was negative when $c = 0$, but it will turn out to be positive for some values of c. Here's a graph.



A little bit of calculus will find the c that maximizes μ, thus giving the best possible return in terms of the almost sure limit of $\frac{S_n}{n}$. The derivative $\frac{d}{dc}\,\mu = -\frac{0.220}{1.4 - 0.4\,c} + \frac{0.180}{0.6 + 0.4\,c}$.
Setting that = 0 and solving gives us $c = 0.75$. As we see in the graph, that is a maximum, and the value of μ at $c = 0.75$ is a positive number, which turns out to be approximately 0.00500836683.

With 75% of her wealth in cash, Rosie's expected wealth is multiplied by a more modest $1.04 - 0.04 \cdot 0.75 = 1.01$ each week: only one percent expected growth per week. But more importantly, she now will have $\frac{S_n}{n}$ approaching this positive value of μ almost surely. In particular, with probability 1 her investment will eventually be worth a lot of money (although it may take a rather long time). Again, the Central Limit Theorem can help quantify that: as mentioned $\mu = 0.55 \log(1.4 - 0.4\,c) + 0.45 \log(0.6 + 0.4\,c)$, while σ turns out to be $\left(\log(1.4 - 0.4 \cdot c) - \log(0.6 + 0.4 \cdot c)\right)\sqrt{0.55\,(1 - 0.55)}$.

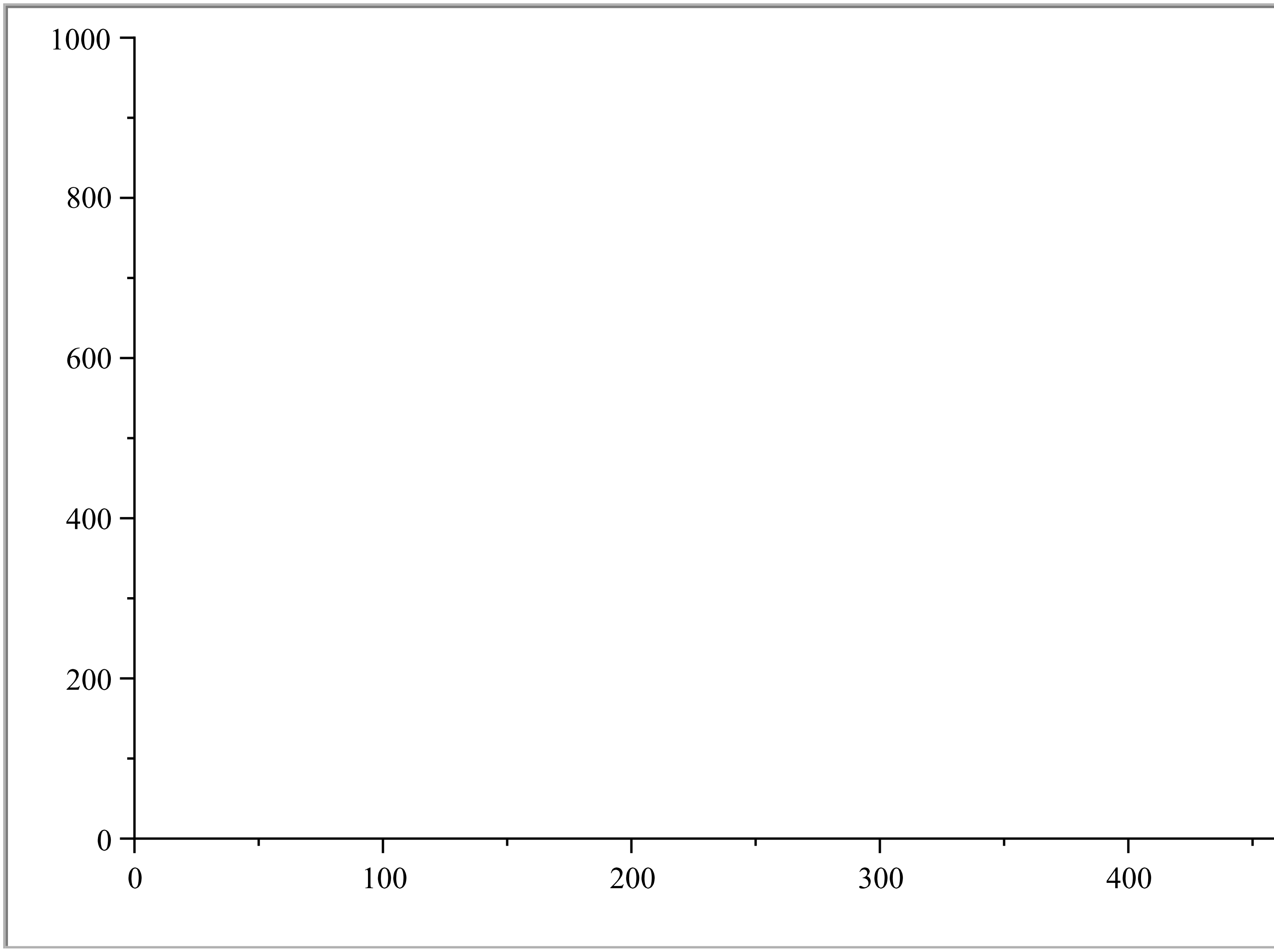
Here is a simulation of Rosie's results with this more prudent plan. The cash fraction c can be adjusted to anything from 0 to 1.

Initial investment (including cash)	1000
Cash fraction	.75
Number of simulations	5
Duration in weeks	500

Linear Mode

Logarithmic Mode

Go



In the table below, you can enter the number of weeks, initial investment, cash fraction and an investment target. When you press the Go button, Maple will calculate the probability that after that many weeks Rosie's investment will be worth at least as much as the target. This is done in two ways: "Probability to achieve" is calculated using the Binomial distribution for the number of times the investment gains in the given time period, and "Normal approximation" is calculated using a normal distribution (with continuity correction).

Duration in weeks	500
Initial investment (including cash)	1000
Cash fraction	.75
Investment target	1000

Go

Probability to achieve	
Normal approximation	

Suggestions for further investigation

1. Derive the formulas for the standard deviation σ, both in the original case and with a cash fraction *c*.
2. Using the Central Limit Theorem, what will be the approximate median value of Rosie's worth after *n* weeks, i.e. a target value that she has approximately probability 0.5 of achieving? Do this both in the original case and with a cash fraction of 0.75, and verify this using the tools in this document. What about the first and third quartiles?
3. Using the Central Limit Theorem, approximately how large must *n* be if (in the original case) $P(W_n < W_0) > .98$? Approximately how large must *n* be if (in the case with a cash fraction of 0.75) $P(W < W_0) < .02$? Refine your approximation using the tools in this document.
4. What other strategies might Rosie use to try to avoid losing money? How might these be analysed?
5. Try some other probability distributions, discrete or continuous, for the X_i. Can you find some where a similar effect occurs: $E(W_n)$ grows but $W_n \to 0$ almost surely as $n \to \infty$?