***INVESTIGATING ON THE POWER SPECTRAL DENSITY OF DUFFING'S EQUATION BY EQUIVALENT LINEARIZATION METHOD

By **CO . H . TRAN .**
Faculty of Mathematics & Informatics , University of Natural Sciences – VNU-HCM

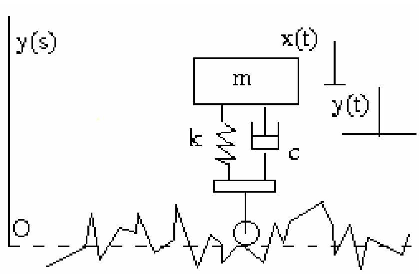
Abstract :
We consider the non-linear random vibration model demonstrated by the Duffing's differential equation : $x''+2\xi\omega_0 x'+\omega_0^2 x+\mu\beta x^3 = f(t)$ *(*)*

The stationary random process is f(t) which is satisfied < f(t) > = 0 with the spectral density function Sf (ω) . To find the solution Sx (ω) of (*) we use the equivalent linearization method .

1/. Model Definition :

The non-linear random vibration model includes the mass (m) - dashpot (c) -spring (k) (fig.1) . This model moves on the rough surface which is described by the random variable y(s) with the constant velocity v . If we have the relation s = vt and the mass m is also influenced under the non-linear stimulating force $\mu\beta x^3$, then the vibration differential equation of the mass m can be rewritten as :

$$x''+2\xi\omega_0 x'+\omega_0^2 x+\mu\beta x^3 = f(t) \qquad (1.0)$$



(fig . 1)

2/. The equivalent linearization method .

The conditions of the stationary solution and equivalent approximation :

$$x''+2\xi\omega_0 x'+\omega_0{}^2 x+\delta x = f(t) \qquad (2.1)$$

The linear operator : $Q(D) = (D^2 + 2\xi\omega_0 D + \omega_o{}^2 + \delta)$ (2 2)

Substitute D = iω into (2.2) we obtain the frequency response :

$$F(\omega) = -\omega^2 + 2\xi\omega_0 i\omega + \omega_o{}^2 + \delta \qquad (2.3)$$

The impulse response :

$$H(\omega) = \frac{1}{-\omega^2 + 2\xi\omega_0 i\omega + \omega_o{}^2 + \delta} \qquad (2.4)$$

The power spectral density :

$$S_x(\omega) = |H(\omega)|^2 S_f(\omega) = \frac{S_f(\omega)}{(\omega_0^2 - \omega^2 + \delta)^2 + 4\omega_0^2\omega^2\xi^2} \qquad (2.5)$$

Assuming S f (ω) = So : const (white-noise) then we have :

$$R_x(0) = E\{x^2\} = \tfrac{1}{2\pi}\int_{-\infty}^{+\infty} |H(\omega)|^2 S_f(\omega)d\omega = \tfrac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{S_o}{(\omega_0^2 - \omega^2 + \delta\ )^2 + 4\omega_0^2\omega^2\xi^2} d\omega \qquad (2.6)$$

By altering : $\rho = 2\xi\omega o \ ; \ \gamma = \omega_0^2 + \delta$ and choosing S f (ω) = So = 1 (to simplify the next algorithm) , we take into account the integral expression :

$$\int_{-\infty}^{\infty} \frac{1}{2}\frac{1}{\pi\left((\omega 0^2 - \omega^2 + \delta)^2 + 4\,\omega 0^2\,\omega^2\,\xi^2\right)}\,d\omega = \int_{-\infty}^{\infty} \frac{1}{2}\frac{1}{(\rho^2\,\omega^2 + (\gamma - \omega^2)^2)\,\pi}\,d\omega \qquad (2.7)$$

The function h(z) : > **h(z):=(1/((rho^2*z^2+(Gamma-z^2)^2))/(2*Pi));**

$$h(z) := \frac{1}{2\,(\rho^2\,z^2 + (\Gamma - z^2)^2)\,\pi}$$

And the equation : $eqn := \rho^2\,z^2 + (\Gamma - z^2)^2 = 0$

(2.8)

Roots of (2.8) :

$$cdiem := \frac{\sqrt{-2\rho^2+4\Gamma+2\sqrt{\rho^4-4\rho^2\Gamma}}}{2}, -\frac{\sqrt{-2\rho^2+4\Gamma+2\sqrt{\rho^4-4\rho^2\Gamma}}}{2}, \frac{\sqrt{-2\rho^2+4\Gamma-2\sqrt{\rho^4-4\rho^2\Gamma}}}{2}, -\frac{\sqrt{-2\rho^2+4\Gamma-2\sqrt{\rho^4-4\rho^2\Gamma}}}{2} \quad (2.9)$$

We choose the main value of (2.9) $z1 := \frac{-1}{2} I \sqrt{2\rho^2-4\Gamma-2\sqrt{\rho^4-4\rho^2\Gamma}}$

Use (2.9) to find the residue of h(z) :

```
> simplify(residue(h(z),z=z1));
```

$$\frac{\frac{1}{2} I}{\pi\sqrt{2\rho^2-4\Gamma-2\sqrt{-\rho^2(-\rho^2+4\Gamma)}}\sqrt{-\rho^2(-\rho^2+4\Gamma)}}$$

The formula of $E\{x^2\}$

```
> Ex2:=S[0]*1/(2*Pi)*%;
```

$$Ex2 := -\frac{1}{2}\frac{S_0}{\pi\sqrt{2\rho^2-4\Gamma-2\sqrt{-\rho^2(-\rho^2+4\Gamma)}}\sqrt{-\rho^2(-\rho^2+4\Gamma)}}$$

```
> delta:=3*mu*beta*Ex2;
```

$$\delta := -\frac{3}{2}\frac{\mu\beta S_0}{\pi\sqrt{2\rho^2-4\Gamma-2\sqrt{-\rho^2(-\rho^2+4\Gamma)}}\sqrt{-\rho^2(-\rho^2+4\Gamma)}}$$

```
> delta:=subs(rho=2*omega[0]*psi,delta);
```

$$\delta := -\frac{3}{2}\frac{\mu\beta S_0}{\pi\sqrt{8\omega_0^2\psi^2-4\Gamma-2\sqrt{-4\omega_0^2\psi^2(-4\omega_0^2\psi^2+4\Gamma)}}\sqrt{-4\omega_0^2\psi^2(-4\omega_0^2\psi^2}}$$

```
> deta:=subs(gamma=omega[0]^2+Delta,delta);
```

$$deta := -\frac{3}{2}\frac{\mu\beta S_0}{\pi\sqrt{8\omega_0^2\psi^2-4\Gamma-2\sqrt{-4\omega_0^2\psi^2(-4\omega_0^2\psi^2+4\Gamma)}}\sqrt{-4\omega_0^2\psi^2(-4\omega_0}}$$

```
> eqndelta:=Delta=deta;
```

$$eqndelta := \Delta = -\frac{3}{2}\frac{\mu\beta S_0}{\pi\sqrt{8\omega_0^2\psi^2-4\Gamma-2\sqrt{-4\omega_0^2\psi^2(-4\omega_0^2\psi^2+4\Gamma)}}\sqrt{-4\omega_0^2\psi^2}}$$

$$R_x(0) = E\{x^2\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |H(\omega)|^2 S_f(\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{S_o}{(\omega_0^2 - \omega^2 + \delta\ )^2 + 4\omega_0^2\omega^2\xi^2} d\omega \qquad (2.10)$$

$$E\{xg(x)\} = \int_{-\infty}^{+\infty} x.\mu\beta x^3 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}} dx \qquad (2.11)$$

```
> Int((mu*beta/(sigma*sqrt(2*Pi)))*x^4*exp(-x^2/(2*sigma^2)),x=-infinity..infinity);
```

$$\int_{-\infty}^{\infty} \frac{1}{2} \frac{\mu \, \beta \, \sqrt{2} \, x^4 \, \mathbf{e}^{\left(-1/2 \frac{x^2}{\sigma^2}\right)}}{\sigma \sqrt{\pi}} \, dx \qquad (2.12)$$

```
Exg(x):=int((mu*beta/(sigma*sqrt(2*Pi)))*x^4*exp(-x^2/(2*sigma^2)),x=-infinity..infinity);
```

$$\mathrm{Exg}(x) := \begin{cases} 3\,\mu\,\beta\,\sigma^4\,\mathrm{csgn}(\overline{(\sigma)}) & \mathrm{csgn}(\overline{(\sigma)}^2) = 1 \\ \infty & \textit{otherwise} \end{cases} \qquad (2.13)$$

The coefficient of equivalent linearization : $\delta = \frac{E\{x.g(x)\}}{E\{x^2\}} = \frac{3\mu\beta\sigma^4.c\mathrm{sgn}(\overline{(\sigma)})}{\sigma^2} = 3\mu\beta\sigma^2.c\mathrm{sgn}(\overline{(\sigma)})$

(2.14)

Calculation in details : >

```
eq:=subs(psi=1,mu=0.1,beta=0.2,S[0]=1,Gamma=omega[0]^2+Delta,eqndelta);eq:=subs(omega[0]=0.5,eq);
```

$$eq := \Delta = -\frac{0.03000000000}{\pi\sqrt{4\,\omega_0^2 - 4\,\Delta - 2\sqrt{-16\,\omega_0^2\,\Delta}}\,\sqrt{-16\,\omega_0^2\,\Delta}}$$

$$eq := \Delta = -\frac{0.03000000000}{\pi\sqrt{1.00 - 4\,\Delta - 2\sqrt{-4.00\,\Delta}}\,\sqrt{-4.00\,\Delta}}$$

```
nodelta:=solve(eq,Delta);
```

$$nodelta := -0.2675483392, -0.2286403831, -0.03981894531$$

The Duffing's equation can be approximated in the linear form with the values of **nodelta** :

$$x''+2\xi\omega_0 x'+(\omega_0{}^2+\delta)x=f(t) \qquad (2.15)$$

The investigation on components of the Duffing's differential equation will be calculated by other methods of linear random vibration , and we can obtain the corresponding approximate values in the meaning of minimum variance .

3/. Parameters – Solution of the equivalent differential equation .

The graph of Duffing's differential equation (non-linear random) : >

```
D(D(x))(t)+2*psi*omega*D(x)(t)+(omega^2)*x(t)+mu*beta*(x(t)^3)=x^3;ps
i:=1;omega:=0.5;mu:=0.1;beta:=0.2;
```

$$(D^{(2)})(x)(t)+1.0\,\psi\,D(x)(t)+0.25\,x(t)+0.02\,x(t)^3=x^3$$

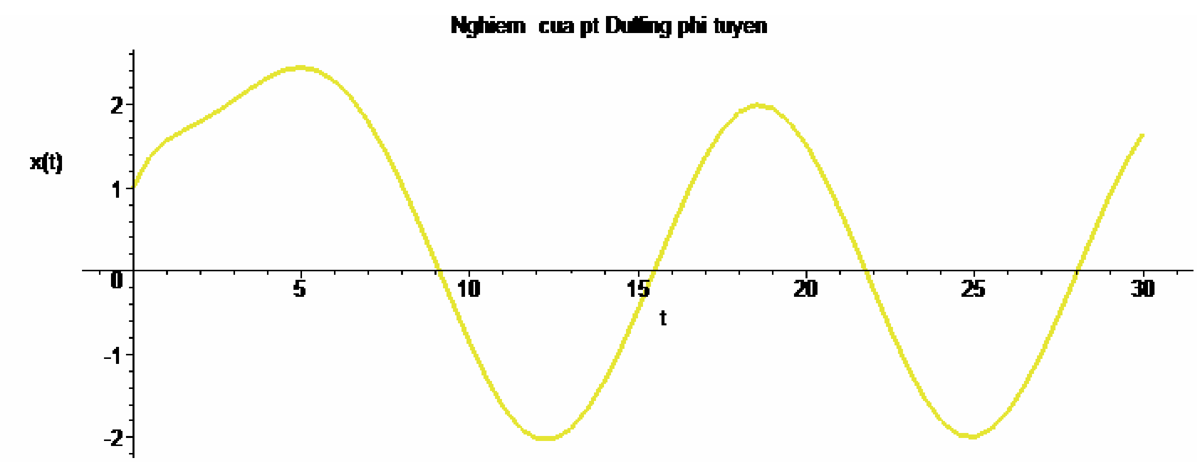
$$\psi := 1$$

$$\omega := 0.5$$

$$\mu := 0.1$$

$$\beta := 0.2$$

```
DEplot({D(D(x))(t)+2*psi*omega*D(x)(t)+(omega^2)*x(t)+mu*beta*(x(t)^3
)=sin(omega*t)},{x(t)},t=0..30,[[x(0)=1,D(x)(0)=1]],stepsize=0.5,titl
e=`Nghiem  cua pt Duffing phi tuyen`);
```



The graph of Duffing's differential equation (equivalent –linearization random) :>

```
D(D(x))(t)+2*psi*omega*D(x)(t)+((omega^2)+delta)*(x(t))=sin(omega*t);
psi:=1;omega:=0.5;delta:=-.3981894531e-1;
```

$$(D^{(2)})(x)(t) + 2\,\psi\,\omega\,D(x)(t) + (\omega^2 + \delta)\,x(t) = \sin(\omega\,t)$$

$$\psi := 1$$

$$\omega := 0.5$$

$$\delta := -0.03981894531$$

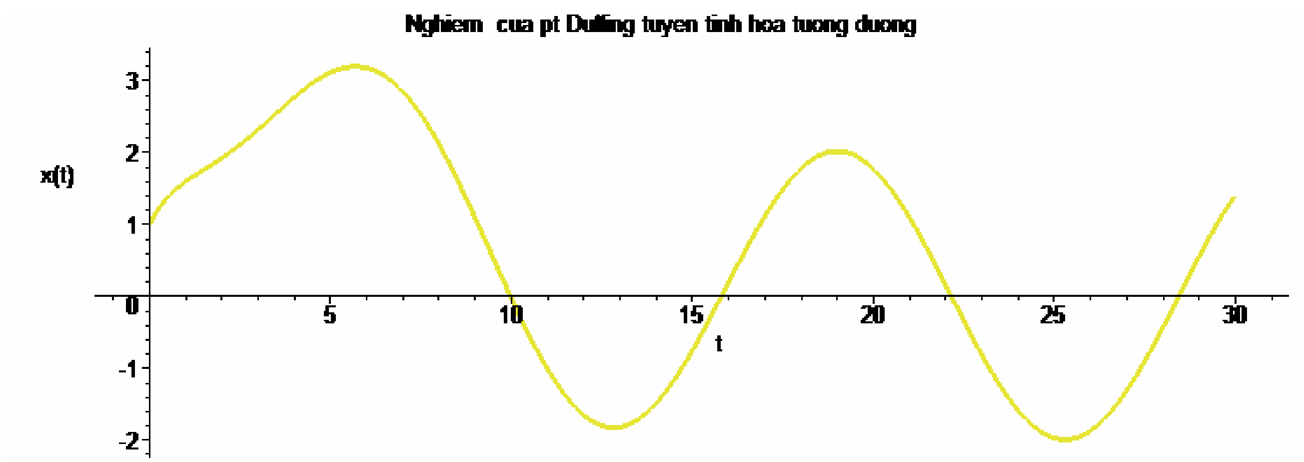
```
DEplot({D(D(x))(t)+2*psi*omega*D(x)(t)+((omega^2)+delta)*(x(t))=sin(o
mega*t)},{x(t)},t=0..30,[[x(0)=1,D(x)(0)=1]],stepsize=0.05,title=`Ngh
iem  cua pt Duffing tuyen tinh hoa tuong duong`);
```

$$(D^{(2)})(x)(t) + 1.0\,D(x)(t) + 0.2101810547\,x(t) = \sin(0.5\,t)$$

$$\psi := 1$$

$$\omega := 0.5$$

$$\delta := -0.03981894531$$



The comparison of two graphical solutions : non-linear and equivalent-linearization .