Classroom Tips and Techniques:

Roles for the Laplace Transform's Shifting Laws

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▼ Introduction

Table 1 lists three shifting laws for the Laplace transform, itself defined by the integral

$$L[f(t)] = \int_0^\infty f(t)e^{-st}\, dt = F(s)$$

First Shifting Law	$L[e^{at}f(t)] = F(s-a)$
Second Shifting Law	$L[f(t-a)\,\text{Heaviside}(t-a)] = e^{-as}F(s)$
Third Shifting Law	$L[g(t)\,\text{Heaviside}(t-a)] = e^{-as}L[g(t+a)]$
Table 1 Three shifting laws for the Laplace transform; in each, $a > 0$	

The first shifting law might be loosely articulated as "multiplication by an exponential in the time domain causes a shift in the transform domain," while the second appears to be its converse: "multiplication by an exponential in the transform domain causes a shift in the time domain." This wonderful symmetry between the first and second shifting laws may be the reason why these two are more prominent in the classroom than the third. In fact, in all my years teaching Laplace transforms at the Rose-Hulman Institute of Technology, the third shifting law was never mandated as part of the syllabus. A statement of this law usually appeared on the Laplace transform "cheat sheet" students were allowed to use on departmental final exams, but it was left to individual instructors to cover it or not.

For students unaware of the third shifting law, the Laplace transform of a function multiplied by $\text{Heaviside}(t-a)$ could be a significant challenge. They have to obtain the transform with the *second* shifting law, a process that typically requires algebraic virtuosity in whose absence the functioning of the law becomes opaque. It would be so much better to teach students to use the third shifting law for this application, and to use the second shifting law for inverting transforms that contain the "second shift flag," namely, the factor e^{-as}.

In this article we explore examples where the "forward" transform of $f(t)\,\text{Heaviside}(t-a)$ is

obtained with the second shifting law, and compare the computational effort to that expended by doing the same transform with the third shifting law. Our conclusion is that students should learn the third shifting law for the "forward" transform, and use the second shifting law only for the inversion of transforms containing an exponential factor.

▼ Symmetry in the Operational Laws

The four basic operational laws for Laplace transforms are listed in Table 2.

$$L[y''] = s^2 Y - sy(0) - y'(0) \qquad L[ty(t)] = -\frac{d}{ds} Y(s)$$

$$L\left[\int_0^t y(x)\, dx\right] = \frac{Y(s)}{s} \qquad\qquad L\left[\frac{y(t)}{t}\right] = \int_s^\infty Y(x)\, dx$$

Table 2   Four operational laws for the Laplace transform

To within a detail for each law, each law can be articulated succinctly and symmetrically, as seen in Table 3.

Time Domain	Transform Domain
Differentiate with respect to t	Multiply by s
Multiply by t	Differentiate with respect to s
Integrate	Divide by s
Divide by t	Integrate with respect to s
Table 3 Symmetry in the four operational laws	

I was fortunate to have been taught the Laplace transform this way in 1965, and I taught it the same way ever since. However, the "devil is in the details," and each law requires attention to an important detail. Thus, when transforming a derivative, you have additional terms, descending in powers of s, and ascending in orders of differentiation. When transforming the product $t\, y(t)$, the differentiation in s requires a minus sign. When transforming an integral, or a division by t, the limits of integration in each case are significant.

Just as for the four basic operational laws, the first and second shifting laws exhibit an interesting symmetry. In the first law, transforming a product containing an exponential factor results in a shift in the transform domain, whereas in the second law, transforming a shift in the time domain results in a product with an exponential factor in the transform domain. But as for the operational laws, the devil is again in the details.

For the first shifting law, the detail would be the sign of a in the factor e^{as}, relative to the direction of

the shift in the transform. For the second shifting law, the detail is the Heaviside factor. Not only must the function $f(t)$ appear with shifted argument, it must also be multiplied by a Heaviside function with the same shifted argument. Applying the second shifting law in the "forward" direction, that is, using it to compute a transform, generally presents great difficulties for many students.

▼ $\mathcal{L}[g(t)\mathbf{Heaviside}(t-a)]$

Computing the Laplace transform of the product $g(t)$ Heaviside$(t-a)$ via the second shifting law requires algebraic virtuosity that must be acquired long before the student enters the differential equations course where the Laplace transform is encountered. Indeed, this virtuosity only works for a special class of functions, namely, polynomials, the exponential function, the sine and cosine functions, and for the really gifted, the hyperbolic sine and cosine functions.

The second shifting law requires that $g(t)$ be manipulated into the form $f(t-a)$ by writing, and simplifying, $g(t)$ as $g((t-a)+a)=f(t-a)$. These transformations are even more difficult to implement in Maple since Maple automatically simplifies $t-a+a$ to t. In Maple, such manipulations are often best expressed as text.

▼ **Example 1**

The Laplace transform of $(3\,t^3+2\,t^2+4\,t-1)$ Heaviside$(t-2)$ is easily obtained in Maple by accessing the **laplace** command through the Context Menu system. (Select Integral Transforms, then Laplace Transform.)

$$(3\,t^3+2\,t^2+4\,t-1)\ \mathrm{Heaviside}(t-2) \xrightarrow{\ \text{Laplace transform}\ } \frac{e^{-2\,s}\left(39\,s^3+48\,s^2+40\,s+18\right)}{s^4}$$

To obtain this result by the second shifting law, each appearance of t in the cubic polynomial must be changed to $(t-2)+2$ so that this polynomial becomes

$$3((t-2)+2)^3+2((t-2)+2)^2+4((t-2)+2)-1$$

and then, after great pains, $3(t-2)^3+20(t-2)^2+48(t-2)+39$. The second shifting law then gives the results in Table 4

$\mathcal{L}\left[3(t-2)^3 \text{ Heaviside}(t-2)\right]$	$= \dfrac{18\, e^{-2s}}{s^4}$
$\mathcal{L}\left[20(t-2)^2 \text{ Heaviside}(t-2)\right]$	$= \dfrac{40\, e^{-2s}}{s^3}$
$\mathcal{L}\left[48(t-2) \text{ Heaviside}(t-2)\right]$	$= \dfrac{48\, e^{-2s}}{s^2}$
$\mathcal{L}\left[39 \text{ Heaviside}(t-2)\right]$	$= \dfrac{39\, e^{-2s}}{s}$

Table 4   Application of second shifting law

Of course, the sum of terms on the right in Table 4 agrees with the transform obtained with Maple's Laplace transform.

To obtain this transform by the third shifting law, simply define the cubic polynomial as the function $g(t)$ and multiply the transform of $g(t+2)$ by the appropriate exponential factor. See Table 5.

• Context Menu: Assign Function	$g(t) = 3\,t^3 + 2\,t^2 + 4\,t - 1 \xrightarrow{\text{assign as function}} g$
• Context Menu: Integral Transforms≻Laplace Transform	$g(t+2) \xrightarrow{\text{Laplace transform}}$ $\dfrac{39\,s^3 + 48\,s^2 + 40\,s + 18}{s^4}$

Table 5   Application of the third shifting law

To complete the calculation, simply multiply the Laplace transform in Table 5 by the exponential factor e^{-2s}.

It would then appear that using the second shifting law to obtain the Laplace transform of a function in the form $g(t) \text{ Heaviside}(t-a)$ requires a great deal of algebraic virtuosity. It is so much easier to use the third shifting law, and to reserve the second shifting law for the inversion of transforms whose form is $e^{-as}F(s)$.

▼ Example 2

The Laplace transform of $e^{-3t} \text{ Heaviside}(t-2)$ is easily obtained in Maple by accessing the **laplace** command through the Context Menu system. (Select Integral Transforms, then Laplace Transform.)

$$e^{-3t} \text{Heaviside}(t - 2) \xrightarrow{\text{Laplace transform}} \frac{e^{-6 - 2s}}{s + 3}$$

To obtain this result by the second shifting law, each appearance of t in the exponential term must be changed to $(t - 2) + 2$ so that this factor becomes

$$e^{-3((t - 2) + 2)} = e^{-3(t - 2) - 6} = e^{-3(t - 2)} e^{-6}$$

The second shifting law then gives

$$\mathcal{L}[e^{-6} e^{-3(t - 2)} \text{Heaviside}(t - 2)] = e^{-6} e^{-2s} \mathcal{L}[e^{-3t}]$$

$$= e^{-6} e^{-2s} \frac{1}{s + 3}$$

Table 6 Application of the second shifting law

The result in Table 6 matches the result given by Maple's Laplace transform.

To obtain this transform by the third shifting law, simply define the exponential factor as the function $g(t)$ and multiply the transform of $g(t + 2)$ by e^{-2s}. See Table 7.

• Context Menu: Unassign	$g \xrightarrow{\text{unassign}} g$
• Context Menu: Assign Function	$g(t) = e^{-3t} \xrightarrow{\text{assign as function}} g$
• Context Menu: Integral Transforms>Laplace Transform	$g(t + 2) \xrightarrow{\text{Laplace transform}} \frac{e^{-6}}{s + 3}$

Table 7 Application of the third shifting law

To complete the calculation, simply multiply the Laplace transform in Table 7 by the exponential factor e^{-2s}.

It would then appear that using the second shifting law to obtain the Laplace transform of a function in the form $g(t) \text{Heaviside}(t - a)$ requires a great deal of algebraic virtuosity. It is so much easier to use the third shifting law, and to reserve the second shifting law for the inversion of transforms whose form is $e^{-as} F(s)$.

▼ Example 3

The Laplace transform of $\sin(3t) \text{Heaviside}(t - 2)$ is easily obtained in Maple by accessing the **laplace** command through the Context Menu system. (Select Integral Transforms, then Laplace

Transform.)

$$\sin(3\,t)\ \text{Heaviside}(t-2) \xrightarrow{\text{Laplace transform}} -\frac{e^{-2\,s}\,(-\sin(6)\,s-3\,\cos(6)\,)}{s^2+9}$$

To obtain this result by the second shifting law, each appearance of t in the sine term must be changed to $(t-2)+2$ so that this trig function becomes

$$\sin(3((t-2)+2)) = \sin(3(t+2)+6) = \sin(3(t-2))\cos(6) + \cos(3(t-2))\sin(6)$$

The second shifting law then gives the results in Table 8.

$\mathcal{L}[\sin(3(t-2))\cos(6)\ \text{Heaviside}(t-2)]$	$= \cos(6)e^{-2\,s}\,\mathcal{L}[\sin(3\,t)]$ $= \cos(6)e^{-2\,s}\,\dfrac{3}{s^2+9}$
$\mathcal{L}[\cos(3(t-2))\sin(6)\ \text{Heaviside}(t-2)]$	$= \sin(6)e^{-2\,s}\,\mathcal{L}[\cos(3\,t)]$ $= \sin(6)e^{-2\,s}\,\dfrac{s}{s^2+9}$

Table 8 Application of the second shifting law

Of course, the sum of terms on the right in Table 8 agrees with the transform obtained with Maple's Laplace transform.

To obtain this transform by the third shifting law, simply define the function $g(t) = \sin(3\,t)$ and multiply the transform of $g(t+2)$ by the appropriate exponential factor. See Table 9.

• Context Menu: Unassign	$g \xrightarrow{\text{unassign}} g$
• Context Menu: Assign Function	$g(t) = \sin(3\,t) \xrightarrow{\text{assign as function}} g$

• Context Menu:
   Integral Transforms>Laplace
   Transform
	$g(t+2) \xrightarrow{\text{Laplace transform}}$
	 $-\dfrac{-\sin(6)\,s-3\,\cos(6)}{s^2+9}$

Table 9 Application of the third shifting law

To complete the calculation, simply multiply the Laplace transform in Table 9 by the exponential factor $e^{-2\,s}$.

It would then appear that using the second shifting law to obtain the Laplace transform of a function in the form $g(t)\ \text{Heaviside}(t-a)$ requires a great deal of algebraic virtuosity. It is so much easier to use the third shifting law, and to reserve the second shifting law for the inversion of transforms whose form is $e^{-a\,s}F(s)$.

▼ $\mathcal{L}^{-1}\!\left[e^{-a\,s}F(s)\right]$

Inverting the Laplace transform $e^{-a\,s}F(s)$ by the second shifting law is far simpler than using this law for computing a transform in the "forward" direction. Once the factor $e^{-a\,s}$ is recognized as the "second shift" flag, that is, as the signal that the second shifting law must be used to obtain the inverse transform, the inversion steps can be articulated by the algorithm given in Table 10.

1. Ignore the factor $e^{-a\,s}$ and invert $F(s)$ to $f(t)$
2. Shift $f(t)$ to $f(t-a)$
3. Multiply $f(t-a)$ by Heaviside$(t-a)$
Table 10 Inversion by second shifting law

Implementing the steps in Table 10 requires no special algebraic skill, no recollection of function-specific properties. The inversion process is as algorithmic with the second shifting law as the transform process is with the third shifting law.

▼ **Example 4**

The inverse of the Laplace transform $\dfrac{s+1}{s^2+1}\,e^{-3\,s}$ is easily obtained in Maple by accessing the **invlaplace** command through the Context Menu system. (Select Integral Transforms, then Inverse Laplace Transform.)

$\dfrac{s+1}{s^2+1}\,\mathrm{e}^{-3\,s} \xrightarrow{\text{inverse Laplace transform}} \text{Heaviside}(t-3)\,(\cos(t-3)+\sin(t-3))$

To obtain this result via first principles, write the rational function as

$$\frac{s}{s^2+1} + \frac{1}{s^2+1}$$

and recognize the first fraction as the transform of $\cos(t)$; the second, as the transform of $\sin(t)$. Sum these terms, replace t with $t-3$ and multiply the resulting expression by Heaviside$(t-3)$. This will result in exactly the expression provided by Maple.

So in summary, use the second shifting law to invert transforms containing an exponential factor, but use the third shifting law to transform functions containing a Heaviside factor.