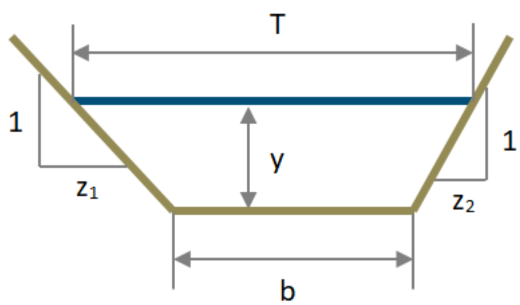
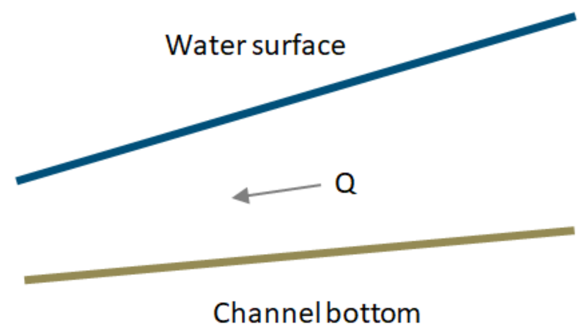
Gradually Varied Flow in a Trapezoidal Channel

Introduction

Water flows along a gently sloped trapezoidal channel with a known initial depth. As the flow progresses along the channel, the water depth eventually reaches a uniform depth that no longer changes with distance along the channel; this is known as the normal depth.





This is known as gradually varied flow. The governing differential equation is

$$\frac{d}{dx} y(x) = \frac{S_0 - S_f}{1 - Fr^2}$$

where S_0 is the channel slope, S_f is the energy gradient (or the rate at which energy is lost via friction) as given by the Manning formula, Fr is the Froude number, and x is the distance along the channel.

The normal depth (i.e. when y(x) no longer changes with x) is given when $S_0 = S_f$.

Given a trapezoidal channel, this application derives formulae for S_f and F_r. These are substituted into the differential equation, which is then numerically solved. Finally, the water surface profile alone the channel is plotted.

Trapezoidal channels have several advantages over rectangular channels. Primarily, the wetted perimeter is small compared to the flow area; this reduces energy losses due to viscous drag, and thus maximizes flow.

Differential Equation for Gradually Varied Flow in a Trapezoidal

Channel

> $restart$

Width of water surface

> $T := b + y(x) \cdot (z_1 + z_2)$

$$T := b + y(x)\,(z_1 + z_2) \tag{2.1}$$

Cross-sectional area of flow

> $A := \frac{y(x)}{2} \cdot (b + T)$

$$A := \frac{y(x)\,(2\,b + y(x)\,(z_1 + z_2))}{2} \tag{2.2}$$

Wetted perimeter

> $P := b + y(x) \cdot \left(\sqrt{1 + z_1^2} + \sqrt{1 + z_2^2}\right)$

$$P := b + y(x)\,\left(\sqrt{z_1^2 + 1} + \sqrt{z_2^2 + 1}\right) \tag{2.3}$$

Hydraulic radius

> $H := \frac{A}{P}$

$$H := \frac{y(x)\,(2\,b + y(x)\,(z_1 + z_2))}{2\left(b + y(x)\,\left(\sqrt{z_1^2 + 1} + \sqrt{z_2^2 + 1}\right)\right)} \tag{2.4}$$

Water velocity

> $v := \frac{Q}{A}$

$$v := \frac{2\,Q}{y(x)\,(2\,b + y(x)\,(z_1 + z_2))} \tag{2.5}$$

Froude number

> $Fr := \frac{v}{\sqrt{g\,y(x)}}$

$$Fr := \frac{2\,Q}{y(x)\,(2\,b + y(x)\,(z_1 + z_2))\,\sqrt{g\,y(x)}} \tag{2.6}$$

Slope of the energy gradient from the Manning equation (this is the rate at which energy is lost through friction)

> $S_f := \left(\frac{n \cdot v}{u \cdot H^{\frac{2}{3}}}\right)^2$

$$S_f := \frac{8\,n^2\,Q^2\,2^{1/3}}{y(x)^2\,\left(2\,b + y(x)\,(z_1 + z_2)\right)^2 u^2 \left(\frac{y(x)\,\left(2\,b + y(x)\,(z_1 + z_2)\right)}{b + y(x)\,\left(\sqrt{z_1^2 + 1} + \sqrt{z_2^2 + 1}\right)}\right)^{4/3}} \quad \textbf{(2.7)}$$

Hence the complete differential equation is

> $de := \frac{d}{dx}\,y(x) = \frac{S_0 - S_f}{1 - Fr^2}$

$$de := \frac{d}{dx}\,y(x) = \frac{S_0 - \frac{8\,n^2\,Q^2\,2^{1/3}}{y(x)^2\,\left(2\,b + y(x)\,(z_1 + z_2)\right)^2 u^2 \left(\frac{y(x)\,\left(2\,b + y(x)\,(z_1 + z_2)\right)}{b + y(x)\,\left(\sqrt{z_1^2 + 1} + \sqrt{z_2^2 + 1}\right)}\right)^{4/3}}}{-\frac{4\,Q^2}{y(x)^3\,\left(2\,b + y(x)\,(z_1 + z_2)\right)^2 g} + 1} \quad \textbf{(2.8)}$$

Parameters

Bottom width

> $b := 3:$

Slope of channel sides

> $z_1 := 2:$
> $z_2 := 3:$

Channel slope

> $S_0 := 0.001:$

Roughness

> $n := 0.025:$

Coefficient in Manning Equation (1 for SI, 1.49 for FPS)

> $u := 1:$

Gravitational constant

> $g := 9.81:$

Flowrate

> $Q := 0.2:$

Maximum channel length

> $L := 1000:$

Water depth at maximum channel length (this will be the boundary condition on the differential equation)

> $y_0 := 0.8:$

The differential equation is reduced to

> de

$$\frac{d}{dx}\,y(x) = \frac{0.001 - \dfrac{0.000200002\,2^{1/3}}{y(x)^2\,(6+5\,y(x))^2\left(\dfrac{y(x)\,(6+5\,y(x))}{3+y(x)\,(\sqrt{5}+\sqrt{10})}\right)^{4/3}}}{-\dfrac{0.01630988787}{y(x)^3\,(6+5\,y(x))^2}+1} \qquad (3.1)$$

Critical Depth and Normal Depth

Normal depth (the constant water depth that the flow reaches at some point along the channel).

> $y_n := fsolve\left(eval\left(S_f = S_0, y(x) = y\right), y = 1\right)$

$$y_n := 0.1667950014 \qquad (4.1)$$

Critical depth

> $y_c := fsolve(subs(\,y(x) = y, Fr = 1), y = 1)$

$$y_c := 0.07380760243 \qquad (4.2)$$

The relative values of y_n and y_c determine the flow profile

Condition	**Flow Profile**
$y_n > y_c$	Mild (M)
$y_n < y_c$	Steep (S)
$y_n = y_c$	Critical (C)

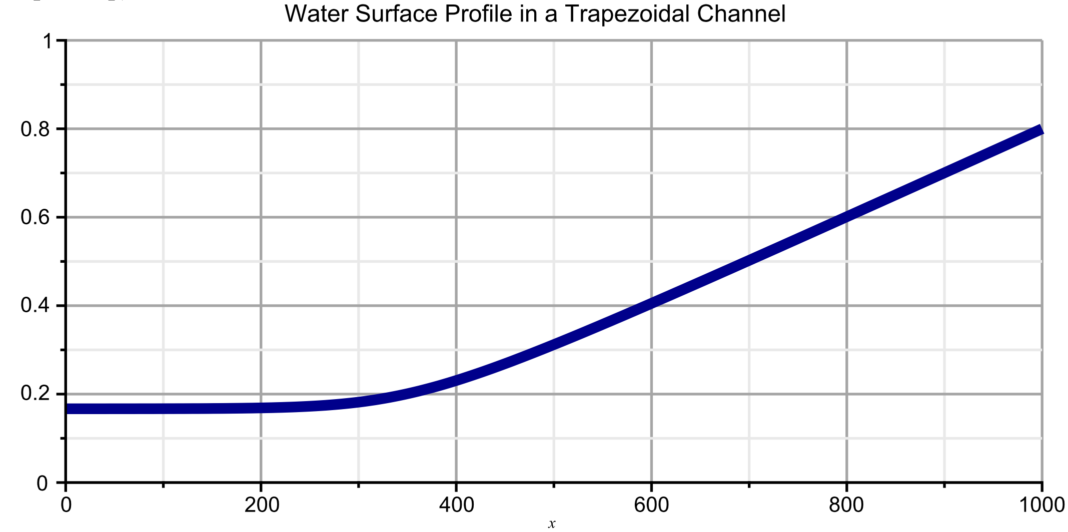
Numerical Solution of the Differential Equation

> $res := dsolve\left(\{de, y(L) = y_0\}, numeric, output = listprocedure\right):$
> $y := subs(res, y(x))$

$$y := \mathbf{proc}(x)\ \ldots\ \mathbf{end\ proc} \qquad (5.1)$$

Plot of the water surface profile along the channel

> $plot(y(x), x = 0..L, gridlines, color = \text{"DarkBlue"}, thickness = 4, size = [800, 400], view = [default, 0..1], title = \text{"Water Surface Profile in a Trapezoidal Channel"}, titlefont = [Arial, 14], axesfont = [Arial])$



Plot of the water surface profile and the slope of the channel bed

> $channel\_slope := x \cdot S_0$:

> $p1 := plot(channel\_slope(x), x = 0..L, legend = \text{"Channel Bed"}, color = \text{"Brown"}, thickness = 8, legendstyle = [font = [Arial]])$:
$p2 := plot(y(x) + channel\_slope, x = 0..L, legend = \text{"Water Profile"}, gridlines, color = \text{"DarkBlue"}, thickness = 4)$:
$plots\text{:-}display(p1, p2, size = [800, 400], view = [default, 0..2], title = \text{"Gradually Varied Flow in a Trapezoidal Channel"}, titlefont = [Arial, 14], axesfont = [Arial], labels = [\text{"Distance along channel"}, \text{"Height"}], labeldirections = [horizontal, vertical], labelfont = [Arial], legendstyle = [location = right, font = [Arial]])$