Modified Atwood Machine

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Abstract

The following is a detailed study of the motion of an unconventional Atwood Machine where one mass is constrained to move along a fixed vertical axis.
The differences with the regular Atwood Machine are :
1- the tension T on the string on either side of the pulley though it is the same, however it is not constant in the present case because of the obliquity of the 2d part of the string.
2- the unique and constant acceleration (a) in the simple machine is replaced in here with two different and variable accelerations whose ratio is however constant.
3- In the simple machine the constant acceleration makes plotting and animation of the system a straightforward procedure according to $s = \frac{1}{2}at^2$***. However in the modified Atwood machine that we present in here the accelerations being variable there is no way to get the displacement as a direct function of time. This seems to make plotting & animation an impossible task. However we were able to devise a trick to overcome this difficulty.***

The following is a **modified Atwood Machine** where the heavier mass m2 is constrained to move along a fixed vertical axis as seen in **Figure-1** below.

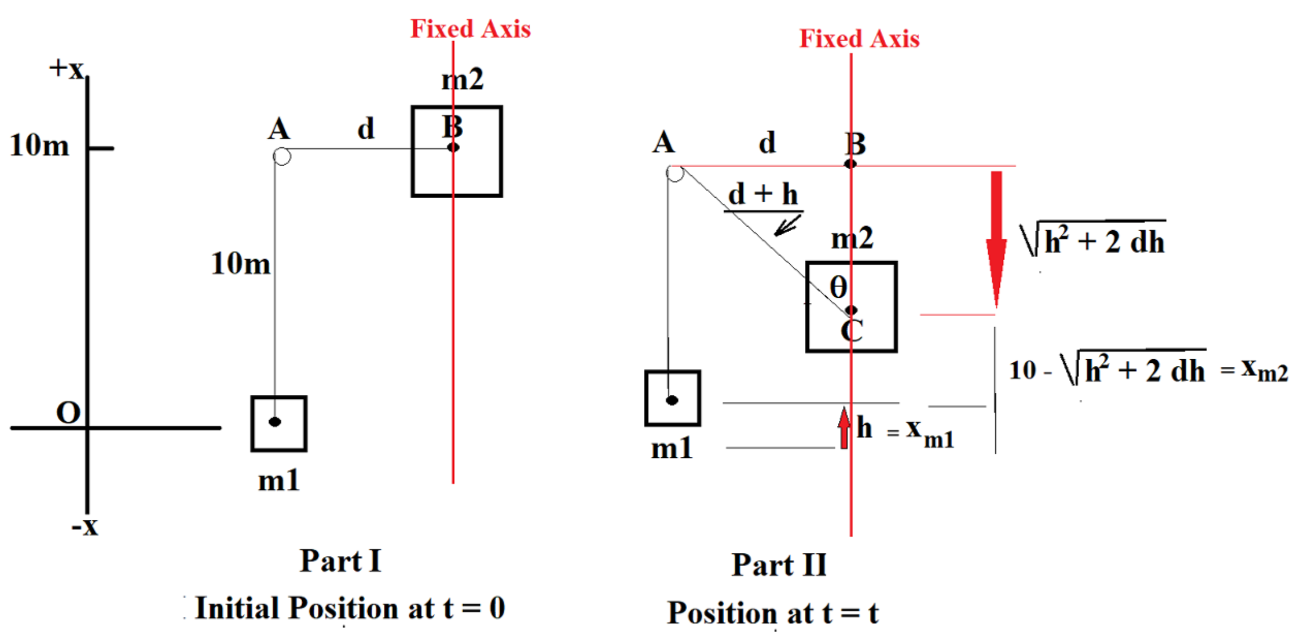


Figure-1

At **t = 0** the lighter mass m1 is at position x = 0 corresponding to **O** origin of the **x-axis with the positive being upward** while mass m2 is initially held in place at a distance **10 meters** above the level of m1. The string attached to m1 & m2 passes over a small pulley of insignificant mass with no friction such that the distance of the Fixed Axis to the pulley is **(d)** meters. Hence the total length of the string is constant and equal to (**10 + d**) meters.
At **t = t** we let go on m2 which is allowed to drop downward in the direction of (**-x**) while m1 moves upward by (**h**) meters which its abscissa x_{m1} from origin **O** i.e. $\boldsymbol{h = x_{m1}}$. When m1 reaches this height, m2 will have dropped from from **B** to **C** such that **AC = h + d**. From right **angle triangle ABC** we get the total drop of m2 as **BC**:

$$BC^2 = (d+h)^2 - d^2 = d^2 + h^2 + 2\,d\,h - d^2 = h^2 + 2\,d\,h,$$

$$BC = \sqrt{h^2 + 2\,d\,h}\,,$$

and its abscissa x_{m2} from origin **O** is

$$\boldsymbol{x_{m2} = 10 - BC = 10 - \sqrt{h^2 + 2\,d\,h}}\ .$$

Since $\boldsymbol{h = x_{m1}}$ then the relation between the abscissa of m1 & m2 along x-axis is:

$$\boldsymbol{x_{m2} = 10 - \sqrt{x_{m1}^{\ 2} + 2\,d\,x_{m1}}}\ . \qquad (1)$$

From the above relation between the abscissa of m1 & m2 we can get the relation between velocities of m1 & m2 by taking the derivative with respect to time of (1):

$$\boldsymbol{\dot{x}_{m2} = -\frac{\dot{x}_{m1}\cdot(x_{m1}+d)}{\sqrt{x_{m1}^{\ 2} + 2\,d\,x_{m1}}}}$$

$$\boldsymbol{\dot{x}_{m2} = \dot{x}_{m1}\cdot\frac{-(x_{m1}+d)}{\sqrt{x_{m1}^{\ 2} + 2\,d\,x_{m1}}}}$$

$$\boldsymbol{\dot{x}_{m2} = \dot{x}_{m1}\cdot\frac{-(h+d)}{\sqrt{h^2 + 2\,d\,h}}}\ . \qquad (2)$$

From this last relation between the velocities we can see that

$$\boldsymbol{|\dot{x}_{m2}| > |\dot{x}_{m1}|}\ ,$$

because $$\boldsymbol{\frac{(h+d)}{\sqrt{h^2+2\,d\,h}} > 1}\ .$$

Note that $$\boldsymbol{\frac{(h+d)}{\sqrt{h^2+2\,d\,h}} = \frac{1}{\cos(\theta)}}$$

hence we have:

$$\boldsymbol{\dot{x}_{m1} = -\dot{x}_{m2}\cdot\cos(\theta)}\ .$$

Therefore, for a given θ, we can take the derivative relative to time of the above to get a similar relation between the accelerations: γ1 & γ2:

$$\boldsymbol{\ddot{x}_{m1} = -\ddot{x}_{m2}\cdot\cos(\theta) \ \rightarrow\ \gamma 1 = -\gamma 2\cdot\cos(\theta)}\ . \qquad (2.1)$$

Note:
Ernest Mach in his
"La Mecanique, French Edition, 1904, p:324"
arrived at the same relation between accelerations as stated above by using a different approch than ours, his method was based on D'Alembert Principle of virtual work.

Getting The Velocities Of m1 & m2 At Any Value Of h

Now to get the value of these velocities for any height **h** of m1 we equate the change in the **Potential Energy (PE)** of the system at any time with the change in its **Kinetic Energy (KE)** since the system, as describe above, is a conservative one.
If we consider the **zero PE** at the origin O then
1- initially at **t = 0** the system has **PE =** $\mathbf{10\cdot m2\cdot g}$ **& no KE**.

2- at **t = t** when m1 has moved **upward** by **h** meters and m2 **downward** by $BC=\sqrt{h^2+2\,d\,h}$ meters then the new **PE** of the system is :

$$\left(10\cdot m2\cdot g - m2\cdot g\cdot\sqrt{h^2+2\,d\,h}+m1\cdot g\cdot h\right),$$

hence the change in **PE** is:

$$10\cdot m2\cdot g-\left(10\cdot m2\cdot g - m2\cdot g\cdot\sqrt{h^2+2\,d\,h}+m1\cdot g\cdot h\right)=m2\cdot g\cdot\sqrt{h^2+2\,d\,h}-m1\cdot g\cdot h$$

while the total **KE** is:

$$\frac{1}{2}\cdot m1\cdot v1^2+\frac{1}{2}\cdot m2\cdot v2^2=\frac{1}{2}\cdot m1\cdot v1^2+\frac{1}{2}\cdot m2\cdot\left(\mathbf{v1}\cdot\frac{-(\mathbf{h}+\mathbf{d})}{\sqrt{\mathbf{h}^2+\mathbf{2\,d\,h}}}\right)^2,$$

where in this last equation we replaced v2 by its value as function of v1 found above in (2).

$$\left(\mathbf{m2\cdot g\cdot\sqrt{h^2+2dh}-m1\cdot g\cdot h}\right)=\frac{\mathbf{1}}{\mathbf{2}}\cdot\mathbf{m1\cdot v1^2}+\frac{\mathbf{1}}{\mathbf{2}}\cdot\mathbf{m2\cdot v1^2}\cdot\left(\frac{\mathbf{-(h+d)}}{\sqrt{\mathbf{h^2+2dh}}}\right)^{\mathbf{2}}\qquad(3)$$

which once solved for v1 gives :

$$\mathbf{v1^2=\frac{2\cdot\left(m2\cdot g\cdot\sqrt{h^2+2\,d\,h}-m1\cdot g\cdot h\right)}{m1+m2\cdot\frac{(h+d)^2}{h^2+2\,d\,h}}}.\qquad(4)$$

Our problem has the fallowing data:
m1 = 1kg, m2 = 5kg, d = 3m, distance from the pulley to m1 = 10m, g = 9.8m/s^2.
We find that when **m1** gets to the pulley i.e. when **h = 10m its speed = 12.88981m/s, with acceleration γ1 = 6.457203001 m/s^2** .
Its free fall from 10m would give it **v =14m/s.**
As for **m2** at t time it will be **12.64911m below the level of the pulley and its speed = 13.24738m/s, with acceleration γ2 = 6.636314487 m/s^2.**

Getting the Time of Ascension of m1 When h = 10m And m2 at 12.64911m Below Pulley Level

To get the time of fall of m2 from **12.64911m height** & that of the ascension of m1 when h = 10m, which must be the same, we need to find first the acceleration of m1 & m2 at that moment when sin(θ) = 3/13 corresponding to θ = 0.23286 radian then we use the following **equations of motion**:

$$T-m1\cdot g=m1\cdot\gamma1\quad\rightarrow T=m1\cdot(g+\gamma1),$$

$$T\cdot\cos(\theta)-m2\cdot g=-m2\cdot\gamma2\quad\rightarrow\quad T=\frac{m2\cdot(g-\gamma2)}{\cos(\theta)}\qquad(5)$$

with equation (2.1) as

$$\boldsymbol{\gamma1=-\gamma2\cdot\cos(\theta)}$$

Once equations of motion are solved for γ1 & γ2 then we use the following equation

$$v=\gamma\cdot t\ \rightarrow\ t=\frac{v}{\gamma},$$

to get the time of ascension of m1 & the time of fall of m2.

We get :

$$\gamma 2 = \frac{g\,(m2 - m1\cos(\theta))}{m1\cdot\cos(\theta)^2 + m2} = 6.636314487,$$

where γ2 is in fact -6.636314487 since it is in the direction of (-x).

and $$\gamma 1 = \frac{g\,(m2 - m1\cos(\theta))}{m1\cdot\cos(\theta)^2 + m2}\cdot\cos(\theta) = 6.457203001,$$

where γ1 is positive since it is in the direction of (+x).
as for the time we have:

$$t1 = \frac{12.88981}{6.457203001} = 1.996190920\ , \qquad t2 = \frac{13.24738}{6.636314487} = 1.996195332\ .$$

As we expected the time for m1 to reach the pulley and the time for m2 to get at **12.64911m below the level of the pulley is exactly the same: t ≈ 2seconds**.

Note about problem getting displacements as function of time

Note that here there is no straightforward approach to get the displacement **s** of m1 or m2 as a direct function of time **t** as is the case in a **conventional Atwood machine** where, once the common and **CONSTANT** acceleration ***a*** of either mass is found from the equations of motion we get the known relation: $s = \frac{1}{2}\cdot a\cdot t^2$ that we can plot for equal increment of time intervals then proceed to the animation.

Here either acceleration γ1 or γ2 is function of time through angle θ which is function of time. **Unfortunately there is no way to get θ as function of time in the present case.**

The other way around this difficulty is to start with equation

$$v = \frac{dx}{dt} = \sqrt{\frac{2\cdot\left(m2\cdot g\cdot\sqrt{x^2 + 2\cdot d\cdot x} - m1\cdot g\cdot x\right)}{m1 + m2\cdot\frac{(x+d)^2}{x^2 + 2\cdot d\cdot x}}}\ ,$$

then integrate from x = 0 to x = h for $0 \le h \le 10$ to get the time t when m1 is at height x = h.
However for plotting purpose **this is not helpful at all** since what we need is to start with a given time then deduce from it the corresponding height.
On the other hand even if the expression above is of some help for our purpose, ***its integration is not possible even numerically!***
To be able to get a **correct and more realistic plotting** we still need the **displacements at equal time intervals.**
The way around this difficulty is explained in the followings:

Our way of getting the displacements of m1 & m2 along with the corresponding time

To get an animation of the system we need the time of descent of m2 which is also that of the ascent of m1.
We solved our problem above, however we could not get the *coordinates of m1 & m2 as function of time directly.* The best we could do is to get the time as we did by :
1- getting 1st angle θ for a given h,
2- getting the speed v1 of m1,
2- solving equations of motion to get γ1 or γ2 knowing their relation in equation (2.1)
3- finally getting the time as t = v/γ.
Using a for.. to.. loop with a great number of cycles (i = 320) we get a list L of the abscissas of m1 & m2 with the corresponding time. In this list L we just extract the smallest interval of time which is found to be around 0.3 second then we take 0.3 as a unit of time and choose from the list L all

multiples of 0.3. With 320 loops we found only 7 multiples of 0.3 since the total time is very short around 2 seconds as we found above. With these 7 intervals of 0.3 second and its multiples we pick from the list the corresponding abscissas of m1 & m2 and make a new list LL with only 7 entries that we plot.
With these data we were able to get the animation of the system .

> *restart* :

> $v1 := \sqrt{\dfrac{2\cdot\left(m2\cdot g\cdot\sqrt{h^2+2\cdot d\cdot h}-m1\cdot g\cdot h\right)}{m1+m2\cdot\dfrac{(h+d)^2}{h^2+2\cdot d\cdot h}}}$; $v2 := v1\cdot\dfrac{-(h+d)}{\sqrt{h^2+2\,d\cdot h}}$

$$v1 := \sqrt{2}\,\sqrt{\frac{m2\,g\,\sqrt{2\,d\,h+h^2}-m1\,g\,h}{m1+\frac{m2\,(h+d)^2}{2\,d\,h+h^2}}}$$

$$v2 := -\frac{\sqrt{2}\,\sqrt{\frac{m2\,g\,\sqrt{2\,d\,h+h^2}-m1\,g\,h}{m1+\frac{m2\,(h+d)^2}{2\,d\,h+h^2}}}\,(h+d)}{\sqrt{2\,d\,h+h^2}} \qquad \textbf{(1)}$$

At h = 2 m

> $g := 9.8;\ d := 3;\ m1 := 1;\ m2 := 5;\ h := 2$

$$g := 9.8$$
$$d := 3$$
$$m1 := 1$$
$$m2 := 5$$
$$h := 2 \qquad \textbf{(2)}$$

> $v1;\ evalf(\%);\ v2;\ evalf(\%);\ \sqrt{2.0\cdot g\cdot 4}$

$$\sqrt{2}\,\sqrt{5.560283688\,\sqrt{16}-2.224113475}$$
$$6.327246047$$
$$-\frac{5}{16}\,\sqrt{2}\,\sqrt{5.560283688\,\sqrt{16}-2.224113475}\,\sqrt{16}$$
$$-7.909057559$$
$$8.854377448 \qquad \textbf{(3)}$$

Note that m1 in free fall from 10m acquires v = sqrt(2*g*10) = -14m/s
Here it gets v = 12.88m/s.
While m2 in free fall from 12.649 will acquire v = -15.74m/s.
Here it gets v = -13.2473m/s

> $h := 10;\ v1;\ evalf(\%);\ v2;\ evalf(\%);\ \sqrt{2.0\cdot g\cdot\sqrt{h^2+2\,d\,h}};\ evalf(\%)$

$$h := 10$$

$$\sqrt{2}\,\sqrt{7.800995025\,\sqrt{160}-15.60199005}$$
$$12.88981451$$
$$-\frac{13}{160}\,\sqrt{2}\,\sqrt{7.800995025\,\sqrt{160}-15.60199005}\,\sqrt{160}$$
$$-13.24738105$$
$$8.854377448\;10^{1/4}$$
$$15.74555710 \qquad (4)$$

Forming our list L with 320 entries and list LL with only 7 entries

We then form our reduced list LL with only 7 time intervals of 0.3 second and its multiples 1*0.3 = 0.3s , 2*0.3 = 0.6s , 3*0.3 = 0.9s , 4*0.3 = 1.2s , 5*0.3 = 1.5s , 6*0.3 = 1.8s , 7*0.3 = 2s we then pick from list L the corresponding abscissas of m1 & m2 and make a new list LL with only 7 entries that we plot.
With these data we were able to get the animation of the system .

> $L := [\,]$; **for** i **to** 320 **do**

$$h := \frac{0.0625}{2}\cdot i\,;\ s1 := h\,;\ s2 := \sqrt{h^2 + 2\,d\cdot h}\,;$$
$$\theta := \arcsin\left(\frac{d}{h+d}\right):$$
$$V1 := evalf(v1):$$
$$\gamma 1 := evalf\left(\frac{g\,(m2 - m1\cdot\cos(\theta))}{m1\cdot\cos(\theta)^2 + m2}\cdot\cos(\theta)\right):$$
$$t := evalf\left(\frac{V1}{\gamma 1}\right):$$
$L := [op(L), [[s1, t], [s2, t]]]$;**od**:

$$L := [\,] \qquad (5)$$

This is the list where the time interval and m1, m2 abscissas are recorded.
The first smallest interval of time is about 0.3s as we see from the first entry:

[[0.03125000000, 0.3048443422], [0.4341388747, 0.3048443422]].

The last entry is:

[[10.00000000, 1.996193322], [12.64911064, 1.996193322]]

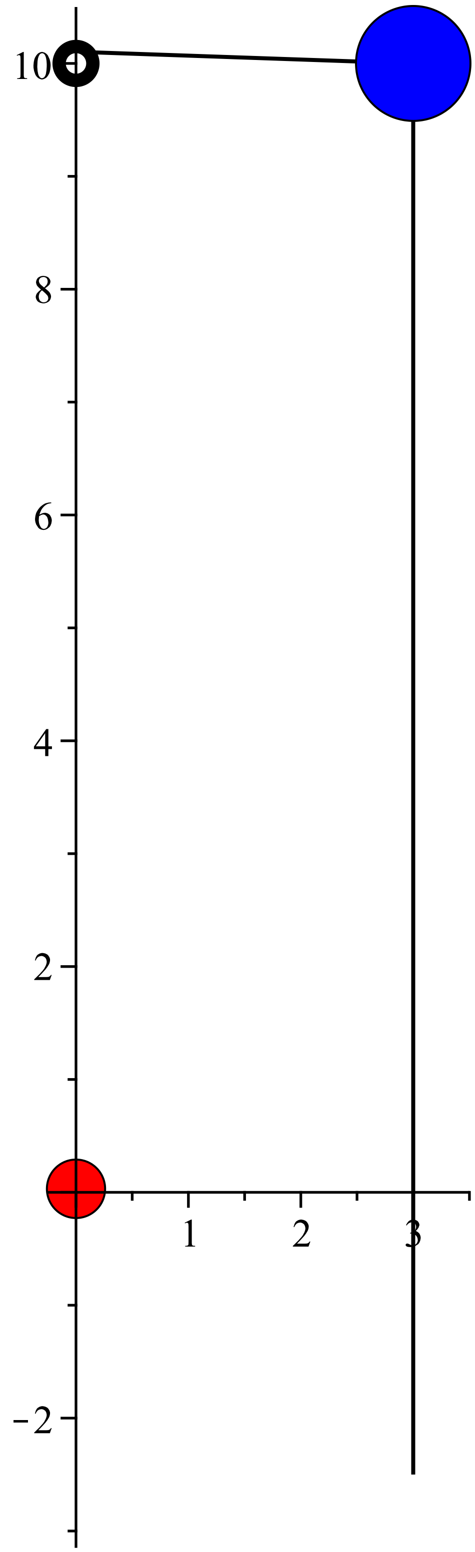
> L:

The reduced list LL with only 7 entries ready for plotting.

> $LL :=$ [[[0.03125000000, 0.3048443422], [0.4341388747, 0.3048443422]], [[0.3750000000, 0.6157379610], [1.546164610, 0.6157379610]], [[1.187500000, 0.9010148733], [2.921498973, 0.9010148733]], [[2.656250000, 1.201307710], [4.795118774, 1.201307710]], [[4.843750000, 1.502440940], [7.247372907, 1.502440940]], [[7.718750000, 1.799497194], [10.29036450, 1.799497194]], [[10.00000000, 1.996193322], [12.64911064, 1.996193322]]]

$LL :=$ [[[0.03125000000, 0.3048443422], [0.4341388747, 0.3048443422]], [[0.3750000000, 0.6157379610], [1.546164610, 0.6157379610]], [[1.187500000, 0.9010148733], [2.921498973, 0.9010148733]], [[2.656250000, 1.201307710], [4.795118774, 1.201307710]], [[4.843750000, 1.502440940], [7.247372907, 1.502440940]], [[7.718750000, 1.799497194], [10.29036450, 1.799497194]], [[10.00000000, 1.996193322], [12.64911064, 1.996193322]]] **(6)**

```
> with(plots) : with(plottools) :
> wood := proc(n)
  local y1, y2, P1, P2, C : global D1, D2 :
  y1 := LL[n][1][1] : y2 := LL[n][2][1] : if y1 < 0.032 then y2 := 0 fi;
  D1 := disk([0, y1], 0.25, color = red) :
  D2 := disk([3, 10 - y2], 0.5, color = blue) : C := circle([0, 10], 0.15, thickness = 5) :
  P1 := polygonplot([[0.1, 10.1], [3, 10 - y2]]) : P2 := polygonplot([[3, -2.5], [3, 10]]) :
  display(D1, D2, P1, P2, C, scaling = constrained) :
  end :
> kvals := seq(k, k = 1 ..7) :
> to_animate := [seq(wood(k), k = kvals)] :
  display(to_animate, insequence = true);
```


10
8
6
4
2
-2
1
2
3

>