Slow Manifold Analysis

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Created for: Topics in Differential Equations
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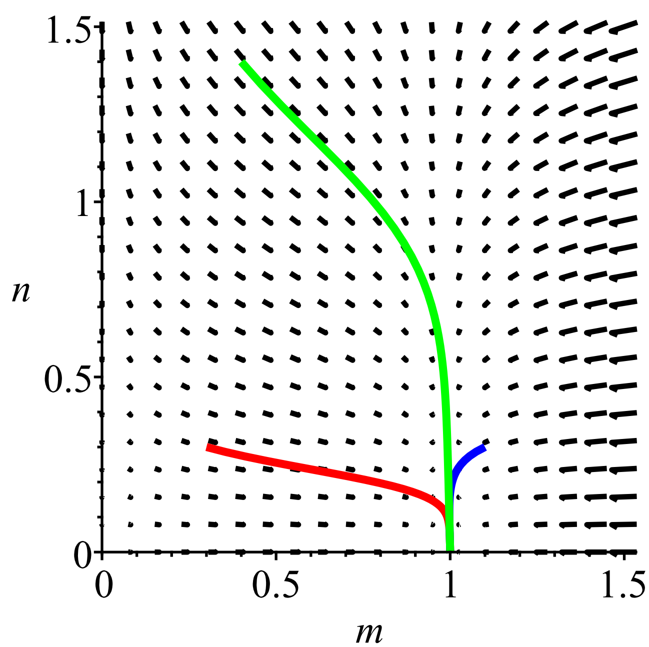
This worksheet goes through the slow manifold analysis following Hek's discussion of the predator prey system. Hek suggests you need to be of order epsilon for this to work.
Epsilon = 0.05 or smaller gives a great match, which epsilon = 0.2 shows that the slow manifold is in fact approximate.

Comments	Calculations
Include necessary packages.	*restart* : *with*(*plots*) : *with*(*DEtools*) :
Define constants.	*epsilon0* := 0.2 : *a0* := *epsilon0* : *d0* := 1 :
Here we define the right hand sides of the ODE in dsolve friendly and field plot friendly form.	$rhsu := u(x)\cdot(1-u(x)) - \frac{\text{epsilon}\cdot a\cdot u(x)\cdot v(x)}{u(x)+d}$ $u(x)\ (1-u(x)) - \frac{\epsilon\, a\, u(x)\ v(x)}{u(x)+d}$ **(1)** $rhsusamp := subs(a = a0, d = d0, \text{epsilon} = epsilon0, rhsu)$ $u(x)\ (1-u(x)) - \frac{0.04\, u(x)\ v(x)}{u(x)+1}$ **(2)** $rhsv := \text{epsilon}\cdot v(x)\cdot\left(\frac{a\cdot u(x)}{u(x)+d} - 1\right)$ $\epsilon\, v(x)\ \left(\frac{a\, u(x)}{u(x)+d} - 1\right)$ **(3)** $rhsvsamp := subs(a = a0, d = d0, \text{epsilon} = epsilon0, rhsv)$ $0.2\, v(x)\ \left(\frac{0.2\, u(x)}{u(x)+1} - 1\right)$ **(4)** $mydeu := diff(u(x), x) = rhsusamp$ $\frac{\mathrm{d}}{\mathrm{d}x}\, u(x) = u(x)\ (1-u(x)) - \frac{0.04\, u(x)\ v(x)}{u(x)+1}$ **(5)**

	mydev := *diff*(*v*(*x*), *x*) = *rhsvsamp* $$\frac{d}{dx}\, v(x) = 0.2\, v(x) \left(\frac{0.2\, u(x)}{u(x)+1} - 1 \right) \quad \textbf{(6)}$$
Solve numerically.	*mysol1* := *dsolve*({*mydeu*, *mydev*, *u*(0) = 0.3, *v*(0) = 0.3}, *numeric*) : *mysol2* := *dsolve*({*mydeu*, *mydev*, *u*(0) = 1.1, *v*(0) = 0.3}, *numeric*) : *mysol3* := *dsolve*({*mydeu*, *mydev*, *u*(0) = 0.4, *v*(0) = 1.4}, *numeric*) :
Create the plots that will be viewed later.	*myplotde1* := *odeplot*(*mysol1*, [*u*(*x*), *v*(*x*)], *x* = 0 ..50, *thickness* = 3, *colour* ='*red*') : *myplotde2* := *odeplot*(*mysol2*, [*u*(*x*), *v*(*x*)], *x* = 0 ..50, *thickness* = 3, *colour* ='*blue*') : *myplotde3* := *odeplot*(*mysol3*, [*u*(*x*), *v*(*x*)], *x* = 0 ..50, *thickness* = 3, *colour* = '*green*') : *myfieldplot* := *fieldplot*([*subs*(*u*(*x*) = *m*, *v*(*x*) = *n*, *rhsusamp*), *subs*(*u*(*x*) = *m*, *v*(*x*) = *n*, *rhsvsamp*)], *m* = 0 ..1.5, *n* = 0 ..1.5, *thickness* = 2) :
Display it all together.	*display*(*myfieldplot*, *myplotde1*, *myplotde2*, *myplotde3*) 
This is the equation for the slow manifold. It assumes that u = p_{ε}(v(x)). The expression is very messy. The ODE for v is used in the expression to take	*sloweqn* := *subs*(*diff*(*v*(*x*), *x*) = *rhsv*, *expand*(*subs*(*u*(*x*) = *p0*(*v*(*x*)) + epsilon·*p1*(*v*(*x*)) + ϵ^2·*p2*(*v*(*x*)), *diff*(*u*(*x*), *x*) = *rhsu*))) **(7)**

care of the chain rule term on the LHS (top of page 358)	$D(p0)(v(x))\,\epsilon\, v(x)\left(\frac{a\,u(x)}{u(x)+d}-1\right)+\epsilon^2 D(p1)(v(x))\,v(x)\left(\frac{a\,u(x)}{u(x)+d}-1\right)+\epsilon^3 D(p2)(v(x))\,v(x)\left(\frac{a\,u(x)}{u(x)+d}-1\right) = p0(v(x))-p0(v(x))^2-2\,p0(v(x))\,\epsilon\,p1(v(x))-2\,p0(v(x))\,\epsilon^2 p2(v(x))+\epsilon\,p1(v(x))-\epsilon^2 p1(v(x))^2-2\,\epsilon^3 p1(v(x))\,p2(v(x))+\epsilon^2 p2(v(x))-\epsilon^4 p2(v(x))^2-\frac{\epsilon\,a\,v(x)\,p0(v(x))}{p0(v(x))+\epsilon\,p1(v(x))+\epsilon^2 p2(v(x))+d}-\frac{\epsilon^2 a\,v(x)\,p1(v(x))}{p0(v(x))+\epsilon\,p1(v(x))+\epsilon^2 p2(v(x))+d}-\frac{\epsilon^3 a\,v(x)\,p2(v(x))}{p0(v(x))+\epsilon\,p1(v(x))+\epsilon^2 p2(v(x))+d}$ **(7)**
This is the leading order solution. Notice there are two and you need to choose the one near the equilibrium point of interest (u = 1).	$sloweqn0 := subs(p0(v(x))=M0, subs(\epsilon=0, sloweqn))$ $0 = -M0^2 + M0$ **(8)** $myp0s := solve(sloweqn0, M0)$ $0, 1$ **(9)** $myp0 := myp0s[2]$ 1 **(10)**
Here is the first correction in which we use the leading order solution.	$sloweqn1 := subs(p1(v(x))=M1, subs(\text{epsilon}=0, diff(subs(p0(v(x))=1, D(p0)(v(x))=0, sloweqn), \text{epsilon})))$ $0 = -M1 - \frac{a\,v(x)}{1+d}$ **(11)** $myp1 := solve(sloweqn1, M1)$ $-\frac{a\,v(x)}{1+d}$ **(12)**
You can extend to second order using the same kind of trick.	$sloweqn2 := subs\left(p2(v(x))=M2, subs\left(\text{epsilon}=0, \frac{1}{2}\cdot diff\left(subs\left(p0(v(x))=1, D(p0)(v(x))=0, p1(v(x))=-\frac{a\cdot v(x)}{1+d}, D(p1)(v(x))=-\frac{a}{1+d}, sloweqn\right), \text{epsilon}\$2\right)\right)\right)$ $-\frac{a\,v(x)\left(\frac{a\,u(x)}{u(x)+d}-1\right)}{1+d} = -M2 - \frac{a^2 v(x)^2}{(1+d)^3}$ **(13)** $myp2 := solve(sloweqn2, M2)$

	$$-\frac{1}{(1+d)^3\,(u(x)+d)}\Big(a\,v(x)\,\big(-u(x)\,a\,d^2 + a\,v(x)\,u(x) + a\,v(x)\,d - 2\,u(x)\,a\,d + u(x)\,d^2 + d^3 - a\,u(x) + 2\,u(x)\,d + 2\,d^2 + u(x) + d\big)\Big) \quad \textbf{(14)}$$
Finally we assemble the slow manifold and plot it.	$linslow := myp0 + \text{epsilon}\cdot subs(v(x) = v, myp1)$ $$1 - \frac{\epsilon\,a\,v}{1+d} \quad \textbf{(15)}$$ $linslowsamp := subs(a = a0, d = d0, \text{epsilon} = epsilon0, linslow)$ $$1 - 0.02000000000\,v \quad \textbf{(16)}$$ $slowplot := plot([linslowsamp, v, v = 0..1.5], thickness = 3, colour = 'brown') :$ $quadslow := myp0 + \text{epsilon}\cdot subs(v(x) = v, myp1) + \varepsilon^2 \cdot subs(v(x) = v, u(x) = 1, myp2)$ $$1 - \frac{\epsilon\,a\,v}{1+d} - \frac{1}{(1+d)^4}\Big(\varepsilon^2\,a\,v\,\big(-a\,d^2 + a\,d\,v + d^3 - 2\,a\,d + a\,v + 3\,d^2 - a + 3\,d + 1\big)\Big) \quad \textbf{(17)}$$ $quadslowsamp := subs(a = a0, d = d0, \text{epsilon} = epsilon0, quadslow)$ $$1 - 0.02000000000\,v - 0.01250000000\,\varepsilon^2\,v\,(7.2 + 0.4\,v) \quad \textbf{(18)}$$ $slowplot2 := plot([quadslowsamp, v, v = 0..1.5], thickness = 3, colour = 'black') :$
We redo the field plot to focus in on the region near u = 1.	$mysol1 := dsolve(\{mydeu, mydev, u(0) = 0.9, v(0) = 0.3\}, numeric) :$ $mysol2 := dsolve(\{mydeu, mydev, u(0) = 1.1, v(0) = 0.3\}, numeric) :$ $mysol3 := dsolve(\{mydeu, mydev, u(0) = 0.9, v(0) = 1.4\}, numeric) :$ $myfieldplot := fieldplot([subs(u(x) = m, v(x) = n, rhsusamp), subs(u(x) = m, v(x) = n, rhsvsamp)], m = 0.9..1.1, n = 0..1.5, thickness = 2) :$ $myplotde1 := odeplot(mysol1, [u(x), v(x)], x = 0..50, thickness = 3, colour = 'red') :$ $myplotde2 := odeplot(mysol2, [u(x), v(x)], x = 0..50, thickness = 3, colour = 'blue') :$ $myplotde3 := odeplot(mysol3, [u(x), v(x)], x = 0..50, thickness = 3, colour = 'green') :$

Display together. To zoom in, right click the picture.	*display*(*myfieldplot*, *myplotde1*, *myplotde2*, *myplotde3*, *slowplot*, *slowplot2*)