Solving ODEs using Maple: An Introduction

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In Maple it is easy to solve a differential equation. Below we show the basic syntax. With this you should be able to use the same basic commands to solve many second-order DEs. First we define the DE and then we use dsolve to find the general solution.

I like to restart just in case Maple remembers something from another worksheet that might cause problems with what we're doing here.

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Introduction to Solving ODES in Maple

General Solutions

We begin by dealing with ODEs with constant coefficients, what you learned to solve in AMATH 250/251. What is nice about these is that we have analytic solutions that we can compare our Maple solutions with.

Process	Calculations	Comments
To the right defines a nonhomogeneous equation.	$ode1 := diff(y(t), t, t) + diff(y(t), t) - 6 \cdot y(t) = 20 \cdot e^{t}$ $\frac{d^2}{dt^2} y(t) + \frac{d}{dt} y(t) - 6\, y(t) = 20\, e^{t}$ **(1.1.1)**	

Then, to find the general solution we use the function dsolve. To get help with this you can type "?dsolve" or use the help bar above. The syntax is to tell Maple the equation we are solving and what variable we are solving for.	$sol1 := dsolve(ode1, y(t))$ $y(t) = e^{-3t}\_C2 + e^{2t}\_C1 - 5e^{t}$ **(1.1.2)**	We see that the solution consists of three components. The first term is a solution to the homogeneous problem, with a constant _C2, the second is a second homogeneous solution, with constant _C1, and the third term is a particular solution. Since the two homogeneous solutions are linearly independent we know that we have a general solution.
If we are feeling skeptical and want to verify that this is in fact a solution we can use odetest.	$odetest(sol1, ode1)$ 0 **(1.1.3)**	It spits out 0, which means that when you plug it in you get zero, so it is infact a solution.
To see what happens when we plug in something that is not a solution try the following.	$odetest(y(t) = t, ode1)$ $1 - 6t - 20e^{t}$ **(1.1.4)**	This is not zero so it is not a solution.

Initial Value Problems

Often we do not only want to find the general solution but we also want to impose conditions. In this course we focus mainly on solving *Initial Value Problems.*

Process	Calculations	Comments
To show how this can be done we first define the initial conditions that we want to impose.	$ics1 := y(0) = 1, D[1](y)(0) = 0$ $y(0) = 1, D(y)(0) = 0$ **(1.2.1)**	
Then we must simply include the ODE and ICs in a list and dsolve will compute the unique solution.	$sol2 := dsolve(\{ode1, ics1\})$ $y(t) = \frac{7}{5}e^{-3t} + \frac{23}{5}e^{2t} - 5e^{t}$ **(1.2.2)**	
We can test whether the solution satisfies the ODE and the ICs.	$odetest(sol2, [ode1, ics1])$ $[0, 0, 0]$ **(1.2.3)**	We get three zeros because our solution satisfies the ode and the two ics.

Boundary Value Problems

Above we solved an *Initial Value Problem.* We can also use Maple to solve *Boundary Value Problems.* Below we solve the same equation but now imposing conditions at two different locations.

Process	Calculations	Comments
First we define the boundary conditions.	$bcs1 := y(0) = 1, y(1) = 2$ $y(0) = 1, y(1) = 2$ **(1.3.1)**	
To solve the BVP we use the same syntax as before.	$sol3 := dsolve(\{ode1, bcs1\})$ $y(t) =$ **(1.3.2)** $-\frac{1}{e^{-3} - e^{2}} (e^{-3t} (-2 + 6 e^{2} - 5 e)) + \frac{1}{e^{-3} - e^{2}} (e^{2t} (6 e^{-3} - 2 - 5 e)) - 5 e^{t}$	
We can test whether the solution is in fact a solution by using odetest.	$odetest(sol3, [ode1, bcs1])$ $[0, 0, 0]$ **(1.3.3)**	As expected, we do in fact have a solution.

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Symbolic versus Numerical Solutions

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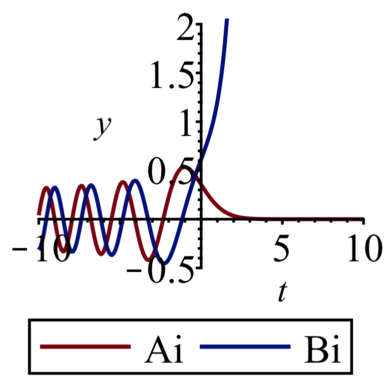
Special DEs and Special Functions

As mentioned in class, there are famous differential equations that we will consider, such as Bessel's and Airy's. These solutions cannot be written in terms of simple functions and, because of their importance, people have defined their solutions to be functions, referred to as *special functions.* This part of the worksheet will find solutions to some DEs and this allows us to look at what these special functions look like.

Airy's Equation

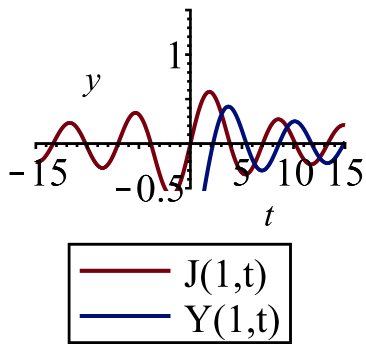
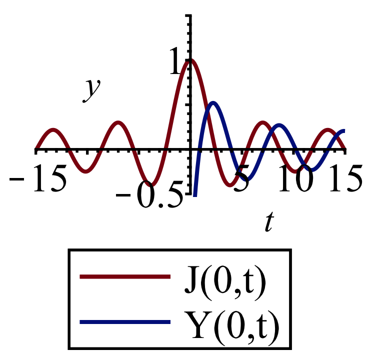
Process	Calculations	Comments
Airy's equation is defined to the right.	$odeairy := diff(y(t), t, t) - t \cdot y(t) = 0$ $\frac{d^2}{dt^2} y(t) - t\, y(t) = 0$ **(3.1.1)**	
We solve this to find the general solution to this homogenenous equation.	$dsolve(odeairy, y(t))$ $y(t) = \_C1\ \mathrm{AiryAi}(t) + \_C2\ \mathrm{AiryBi}(t)$ **(3.1.2)**	Note that there are two solutions that appear, AiryAi and AiryBi. To learn more about them we can use the help command "?AiryAi"
To see what these solutions look like we plot them side by side in the figure to the right using plot.	$plot([\mathrm{AiryAi}(t), \mathrm{AiryBi}(t)], t = -10..10, y = -.5..2, legend = ["Ai", "Bi"])$ 	I included a legend so that we can tell which is which. If we don't specify the bounds on y then we will get huge axes because AiryBi (often referred to as Bi) gets very large, very quickly. Later in the course we will solve this equation using power series and that will help to understand the behaviour of the solutions.
For now we look at the first few terms in AiryAi to see what it looks like. To the right we look at the cubic terms only.	$series(\mathrm{AiryAi}(t), t, 4)$ **(3.1.3)**	This involves the Gamma funciton. To learn what that is we

	$\frac{1}{3}\frac{3^{1/3}}{\Gamma\left(\frac{2}{3}\right)} - \frac{1}{2}\frac{3^{1/6}\,\Gamma\left(\frac{2}{3}\right)}{\pi}t + \frac{1}{18}\frac{3^{1/3}}{\Gamma\left(\frac{2}{3}\right)}t^3 + \mathrm{O}\left(t^4\right)$ **(3.1.3)**	can again ask for help using "?GAMMA". It is the extension of the factorial to the real numbers where Γ(n) = (n-1)!, if n is an integer.

Bessel's Equation of order n

Next, we look at solutions to Bessel's equation that we introduced in class.

Process	Calculations	Comments
Again, we will first define our equation.	$odebessel := diff(y(t), t, t) + \frac{1}{t} \cdot diff(y(t), t) + \left(1 - \frac{n^2}{t^2}\right) \cdot y(t) = 0$ $\frac{d^2}{dt^2} y(t) + \frac{\frac{d}{dt} y(t)}{t} + \left(1 - \frac{n^2}{t^2}\right) y(t) = 0$ **(3.2.1)**	
We solve using dsolve	$dsolve(odebessel, y(t))$ $y(t) = \_C1\ \mathrm{BesselJ}(n, t) + \_C2\ \mathrm{BesselY}(n, t)$ **(3.2.2)**	The two linearly independent solutions can be written as BesselJ(n, t) and BesselY(n,t). **BesselJ(n,t) is the Bessel Function of order n of the first kind. BesselY(n,t) is the Bessel function of order n of the second kind.** We can ask for help using "?BesselJ" if

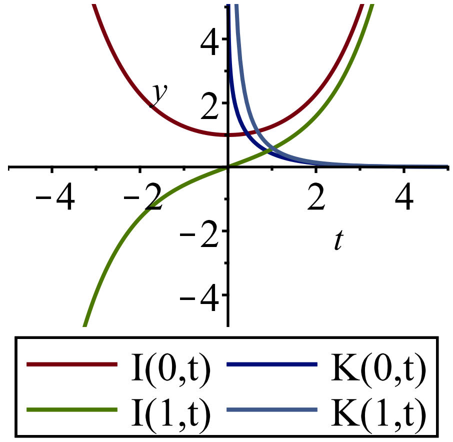
		we need.
We can look at the series representation of a couple of them as we did before.	*series*(BesselJ(0, *t*), *t*, 6) $1-\frac{1}{4}t^2+\frac{1}{64}t^4+O(t^6)$ **(3.2.3)** *series*(BesselJ(1, *t*), *t*, 6) $\frac{1}{2}t-\frac{1}{16}t^3+\frac{1}{384}t^5+O(t^6)$ **(3.2.4)**	Notice that for each integer in the equation, n, we get distinct solutions. How do they compare and how do they differ?
As an easy way to figure this out we can plot them. This shows us the qualitative behaviours. Later, we will find power series solutions that will shed more light on the solution.	*plot*([BesselJ(1, *t*), BesselY(1, *t*)], *t* =-15..15, *y* =-.5..1.5, *legend* = ["J(1,t)", "Y(1,t)"])  *plot*([BesselJ(0, *t*), BesselY(0, *t*)], *t* =-15..15, *y* =-.5..1.5, *legend* = ["J(0,t)", "Y(0,t)"])  	Notice that J(0,t)=1 at t=0 and J(1.t) = 0 at t=0. It turns out that J(n,t)=0 for all n bigger than or equal to n. The solutions seem to oscillate and decay. If you want to see this for larger values of y then you simply need to change the upper boud on t in the plotting command. The functions BesselY(n,t) don't appear for negative values of t. Also, we cannot see where they intersect the y axis. This is beause these functions are actually unbounded at t=0. Another property that we will

		see later on.

Modified Bessel's Equation of order n

Next we look at solutions of the modified Bessel's equation of order n.

Process	Calculations	Comments
This is almost exactly the same equation as we saw previously but in front of the last term we have a -1 instead of a +1. Such a minor difference produces very different solutions.	$odembessel := diff(y(t), t, t) + \frac{1}{t} \cdot diff(y(t), t) - \left(1 + \frac{n^2}{t^2}\right) \cdot y(t) = 0$ $\frac{d^2}{dt^2} y(t) + \frac{\frac{d}{dt} y(t)}{t} - \left(1 + \frac{n^2}{t^2}\right) y(t) = 0$ **(3.3.1)**	
As usual we solve this with dsolve.	$dsolve(odembessel, y(t))$ $y(t) = \_C1\ \mathrm{BesselI}(n, t) + \_C2\ \mathrm{BesselK}(n, t)$ **(3.3.2)**	The two solutions are: **BesselI(n,t), the n-th order Modified Bessel function of the first kind. BesselK(n,t), the n-th order Modified Bessel function of the second kind.**
We plot these solutions to see their behaviours.	*plot*([BesselI(0, *t*), BesselK(0, *t*), BesselI(1, *t*), BesselK(1, *t*)], *t*=-5 ..5, *y*=-5 ..5, *legend*= ["I(0,t)", "K(0,t)", "I(1,t)", "K(1,t)"])	



Finally we will look at series representations of a few of them.

series(BesselI(0, t), t, 6)

$$1+\frac{1}{4}t^{2}+\frac{1}{64}t^{4}+O(t^{6}) \qquad \textbf{(3.3.3)}$$

series(BesselI(1, t), t, 6)

$$\frac{1}{2}t+\frac{1}{16}t^{3}+\frac{1}{384}t^{5}+O(t^{6}) \qquad \textbf{(3.3.4)}$$

series(BesselK(0, t), t, 6)

$$\ln(2)-\ln(t)-\gamma+\left(\frac{1}{4}\ln(2)-\frac{1}{4}\ln(t)+\frac{1}{4}-\frac{1}{4}\gamma\right)t^{2}+\left(\frac{1}{64}\ln(2)-\frac{1}{64}\ln(t)+\frac{3}{128}-\frac{1}{64}\gamma\right)t^{4}+O(t^{6}) \qquad \textbf{(3.3.5)}$$

series(BesselK(1, t), t, 6)

$$t^{-1}+\left(-\frac{1}{2}\ln(2)+\frac{1}{2}\ln(t)+\frac{1}{2}\gamma-\frac{1}{4}\right)t+\left(-\frac{1}{16}\ln(2)+\frac{1}{16}\ln(t)-\frac{5}{64}+\frac{1}{16}\gamma\right)t^{3}+\left(-\frac{1}{384}\ln(2)+\frac{1}{384}\ln(t)-\frac{5}{1152}+\frac{1}{384}\gamma\right)t^{5}+O(t^{6}) \qquad \textbf{(3.3.6)}$$

Notice that we see in the last two exapnsions that there are ln(t) and 1/t, both of which are singular at t=0. This is why these functions blow up at the origin.

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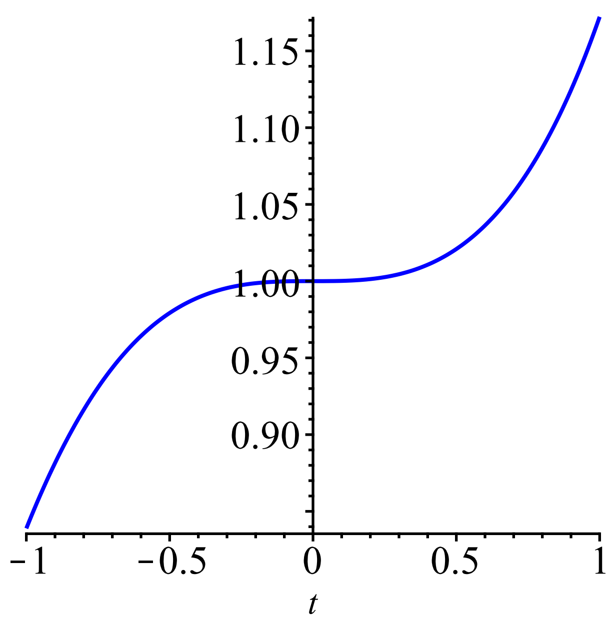
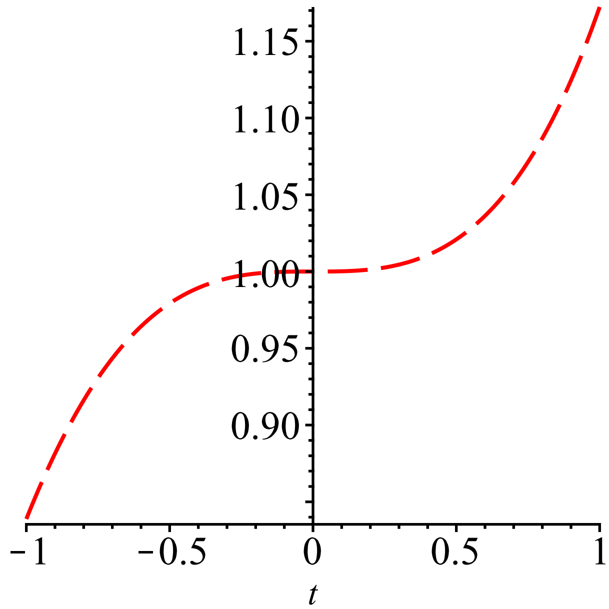
Power Series Solutions to DEs

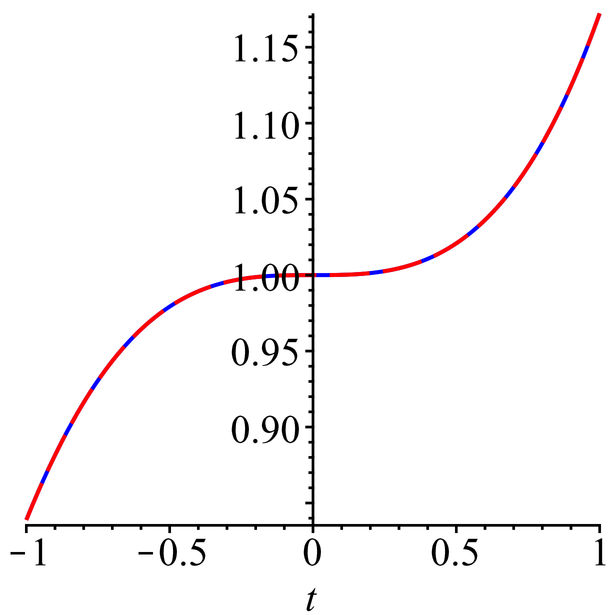
Maple also allows us to find series solutions with only including a finite number of terms in the power series solution.

Process	Calculations	Comments
To illustrate this let's start with a simple equation whose solution we can compute easily.	$odel := diff(y(t), t) = y(t)$ $\frac{d}{dt} y(t) = y(t)$ **(4.1)**	
We use the same syntax as before but now we specify that we want a series solution. The default is 6 terms but we can change that to whatever we would like.	$sol1 := dsolve(\{ode1, y(0) = 1\}, y(t), series)$ $y(t) = 1 + t + \frac{1}{2} t^2 + \frac{1}{6} t^3 + \frac{1}{24} t^4 + \frac{1}{120} t^5 + O(t^6)$ **(4.2)**	
If we want to change the order we must specify Order before we do the above.	$Order := 10 :$ $sol2 := dsolve(\{ode1, y(0) = 1\}, y(t), series)$ $y(t) = 1 + t + \frac{1}{2} t^2 + \frac{1}{6} t^3 + \frac{1}{24} t^4 + \frac{1}{120} t^5 + \frac{1}{720} t^6 + \frac{1}{5040} t^7 + \frac{1}{40320} t^8 + \frac{1}{362880} t^9 + O(t^{10})$ **(4.3)**	Observe that the first six non-zero terms are exactly the same.
It should be clear that this is getting closer and closer to the series expansion of the exponential function.	$series(e^t, t, 10)$ $1 + t + \frac{1}{2} t^2 + \frac{1}{6} t^3 + \frac{1}{24} t^4 + \frac{1}{120} t^5 + \frac{1}{720} t^6 + \frac{1}{5040} t^7 + \frac{1}{40320} t^8 + \frac{1}{362880} t^9 + O(t^{10})$ **(4.4)**	
Now we can apply to the Airy function to see how to solve for	$odeairy := diff(y(t), t, t) - t \cdot y(t) = 0$ **(4.5)**	

the first few components of the series solution.	$\frac{d^2}{dt^2} y(t) - t\,y(t) = 0$ **(4.5)**	
First we pick the initial conditions to be something simple so we can obtain the series expansions for that particular function.	$Order := 8$ 8 **(4.6)** $soln3 := dsolve(\{odeairy, y(0) = 1, \mathrm{D}[1](y)(0) = 0\}, y(t), series)$ $y(t) = 1 + \frac{1}{6} t^3 + \frac{1}{180} t^6 + \mathrm{O}(t^8)$ **(4.7)** $\xrightarrow{\text{at 10 digits}}$ $y(t) = 1. + 0.1666666667\, t^3 + 0.005555555556\, t^6 + \mathrm{O}(t^8)$ **(4.8)**	
We can compare this with the complete solution.	$soln4 := dsolve(\{odeairy, y(0) = 1, \mathrm{D}[1](y)(0) = 0\}, y(t))$ $y(t) = \frac{1}{2} 3^{2/3} \Gamma\left(\frac{2}{3}\right) \mathrm{AiryAi}(t) + \frac{1}{2} \Gamma\left(\frac{2}{3}\right) 3^{1/6} \mathrm{AiryBi}(t)$ **(4.9)**	
We can compare these two solutions in a plot. Here I choose to define the plots seperately and then display them together.	$p1 := plot(convert(rhs(soln3), polynom), t = -1..1, colour = blue)$  $p2 := plot(rhs(soln4), t = -1..1, colour = red, linestyle = dash)$	We see the solid red and dashed blue are almost identical.

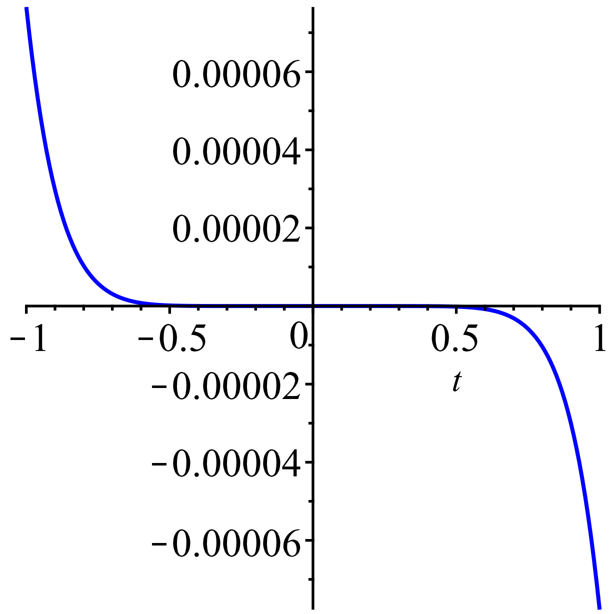


plots[display]([p1, p2])



It is also easy to plot their difference.

plot([convert(rhs(soln3), polynom) − rhs(soln4)], t = −1 ..1, colour = blue)


0.00006
0.00004
0.00002
-1
-0.5
0
0.5
1
t
-0.00002
-0.00004
-0.00006