Delay Differential Equations in Maple

Allan Wittkopf
Maplesoft

Delay Example

Modeling simple harmonic motion with lag:

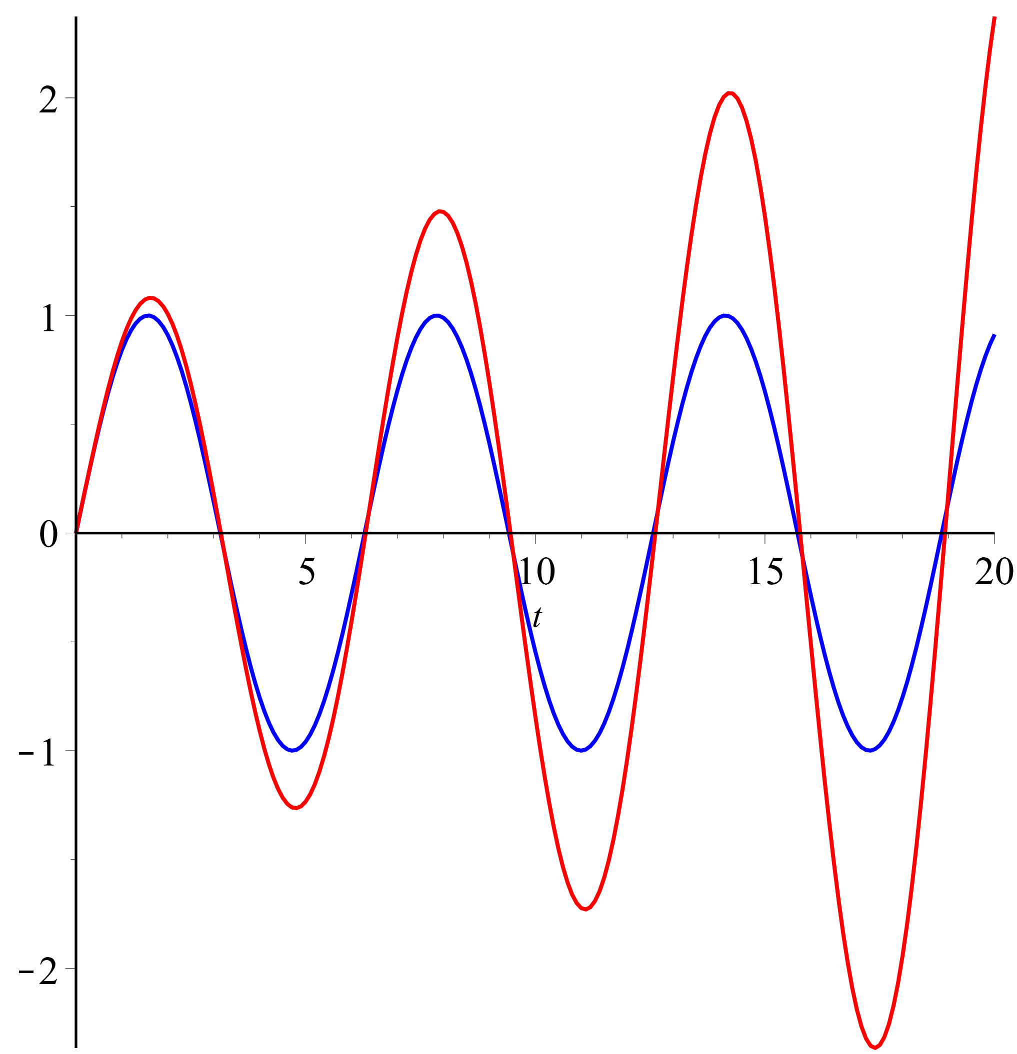
> $ddesys := \left\{ \frac{d^2}{dt^2} x_1(t) + x_1(t - \tau_1) = 0, x_1(0) = 0, D(x_1)(0) = 1, \frac{d^2}{dt^2} x_2(t) + x_2(t - \tau_2) = 0, x_2(0) = 0, D(x_2)(0) = 1 \right\}$

$$ddesys := \left\{ \frac{d^2}{dt^2} x_1(t) + x_1(t - \tau_1) = 0, \frac{d^2}{dt^2} x_2(t) + x_2(t - \tau_2) = 0, x_1(0) = 0, x_2(0) = 0, D(x_1)(0) = 1, D(x_2)(0) = 1 \right\} \quad \textbf{(1.1)}$$

> $dsn := dsolve(eval(ddesys, \{\tau_1 = 0.0, \tau_2 = 0.1\}), numeric)$

$dsn :=$ **proc**(x_rkf45_dae) ... **end proc** **(1.2)**

> $plots[odeplot](dsn, [[t, x_1(t), color = blue], [t, x_2(t), color = red]], 0..20, labels = [t, ""])$



Compare to growth:

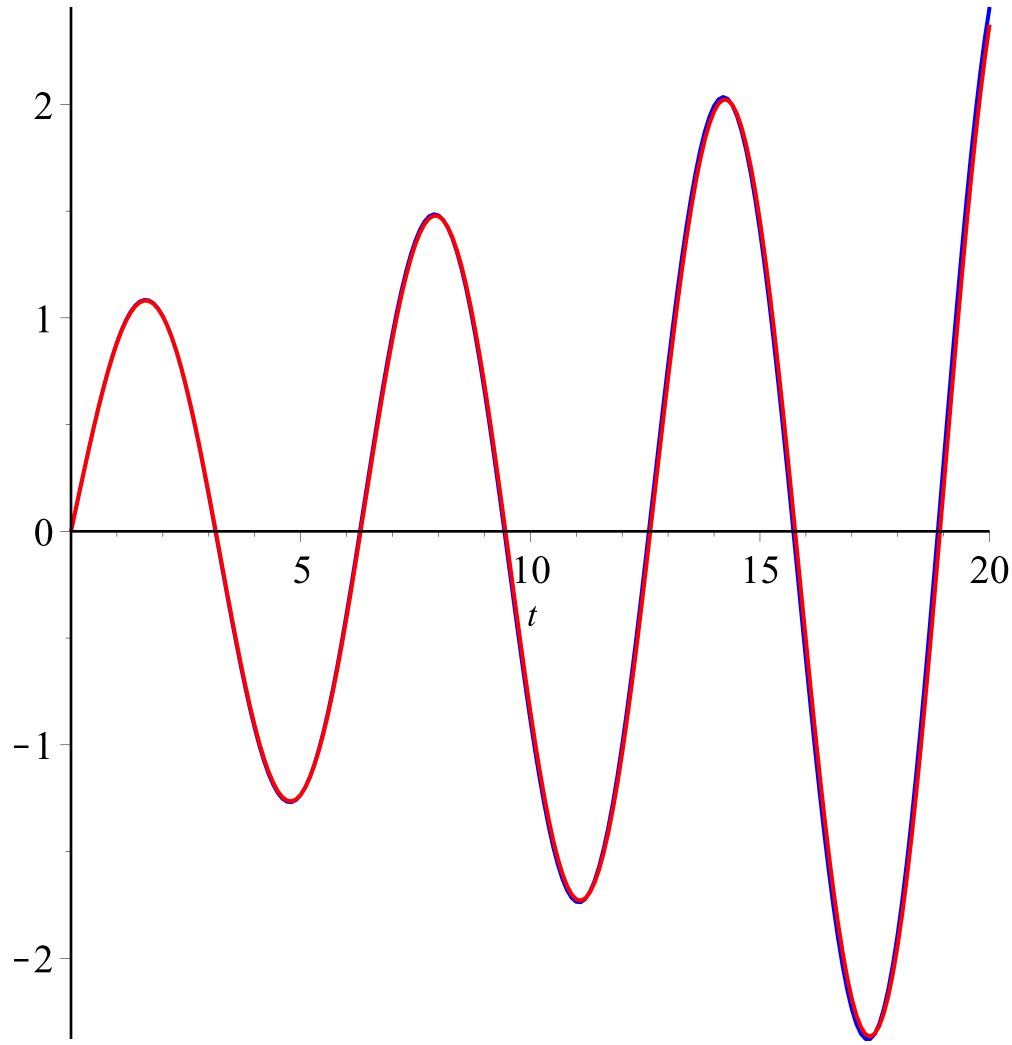
> $ddesys := \left\{ \frac{d^2}{dt^2} x_1(t) - \tau_1 \frac{d}{dt} x_1(t) + x_1(t) = 0, x_1(0) = 0, D(x_1)(0) = 1, \frac{d^2}{dt^2} x_2(t) + x_2(t - \tau_2) = 0, x_2(0) = 0, D(x_2)(0) = 1 \right\}$

$$ddesys := \left\{ \frac{d^2}{dt^2} x_2(t) + x_2(t - \tau_2) = 0, \frac{d^2}{dt^2} x_1(t) - \tau_1 \left(\frac{d}{dt} x_1(t) \right) + x_1(t) = 0, x_1(0) = 0, x_2(0) = 0, D(x_1)(0) = 1, D(x_2)(0) = 1 \right\} \quad \textbf{(1.3)}$$

> $dsn := dsolve(eval(ddesys, \{\tau_1 = 0.1, \tau_2 = 0.1\}), numeric)$

$dsn :=$ **proc**(x_rkf45_dae) ... **end proc** **(1.4)**

> $plots[odeplot](dsn, [[t, x_1(t), color = blue], [t, x_2(t), color = red]], 0..20, labels = [t, ""])$



Implementation Details

- **Mechanism**: Natural C[1] interpolant: rkf45, rosenbrock, ck45
- **Initial Conditions**: Use constant assumption
- **Delay storage**: fixed ('delaypts' option)
- **Variable delay**: supported (use 'delaymax' option)
- **Small delays**: extrapolation
- **Derivatives of delay**: 1st order only (C[1] interpolant)

Preadator-Prey Model

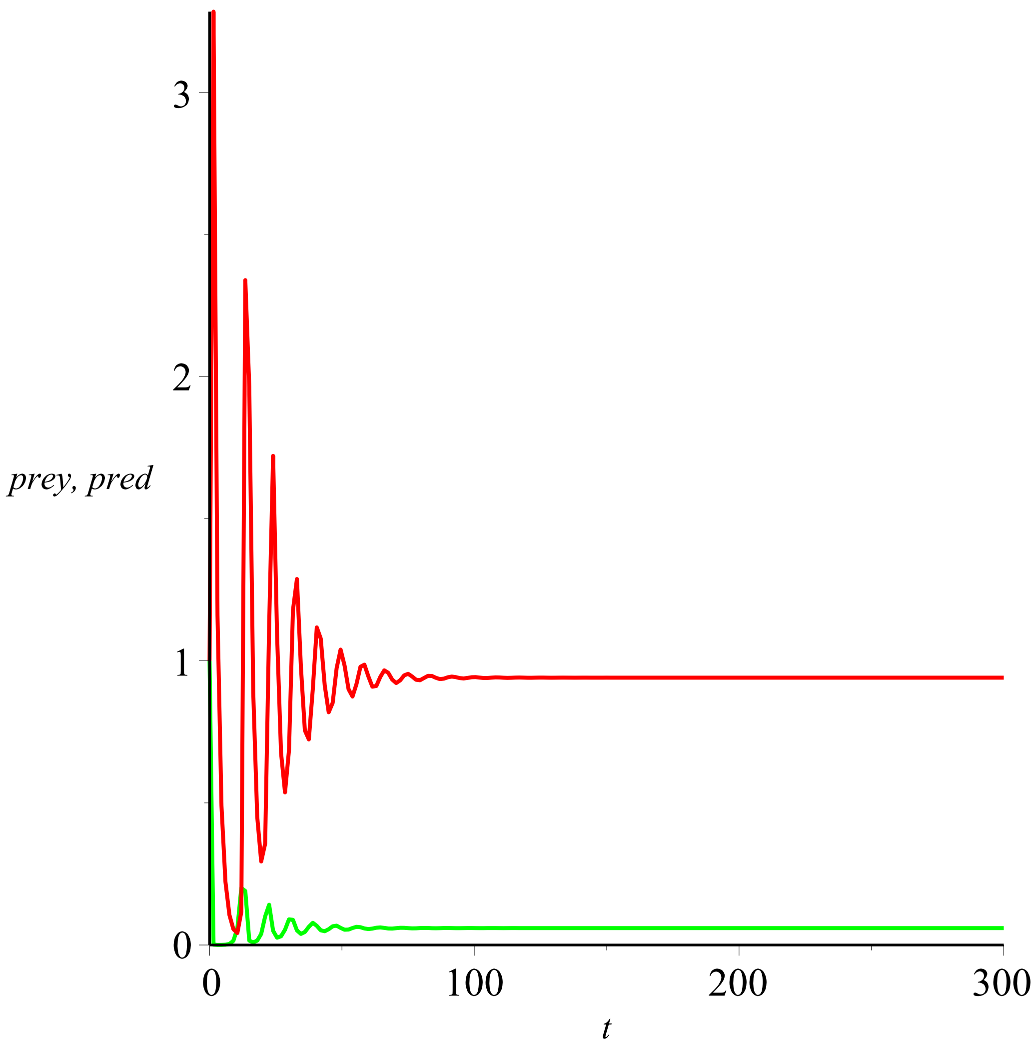
Adapted from Hutchinson model: delay accomodates differences in resource consumption between young and adult members of population:

> $ddesys := \left\{ \frac{\mathrm{d}}{\mathrm{d}\,t}\, prey(t) = prey(t)\ (1 - prey(t)) - pred(t)\ prey(t), \right.$

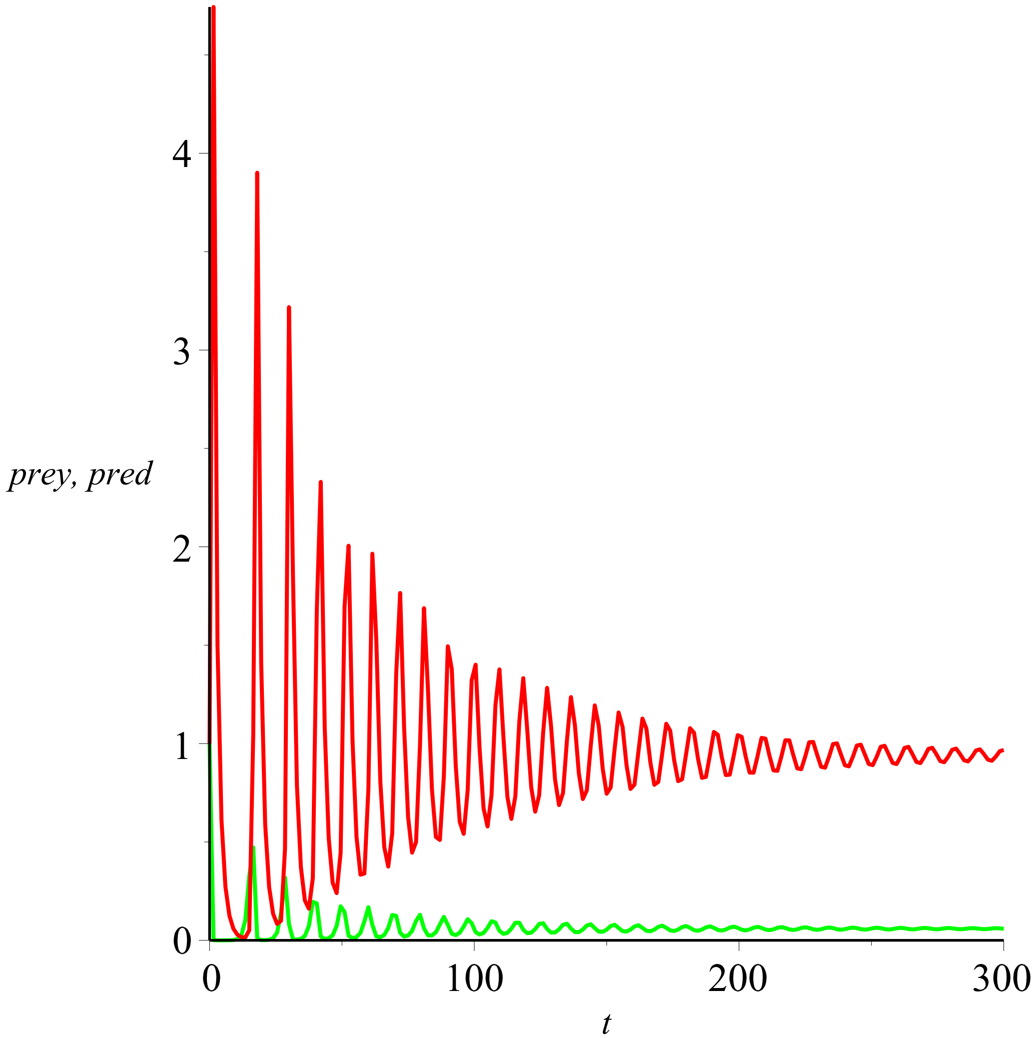
$$\frac{d}{dt}\, pred(t) = 10\, pred(t-\tau)\, prey(t-\tau) - \frac{pred(t)}{2} - \frac{pred(t)^2}{10},$$
$$prey(0) = 1, pred(0) = 1\Big\}$$

$$ddesys := \left\{ \frac{d}{dt}\, pred(t) = 10\, pred(t-\tau)\, prey(t-\tau) - \frac{1}{2}\, pred(t) - \frac{1}{10}\, pred(t)^2, \right. \quad \textbf{(3.1)}$$
$$\left. \frac{d}{dt}\, prey(t) = prey(t)\,(1 - prey(t)) - pred(t)\, prey(t), pred(0) = 1, prey(0) = 1 \right\}$$

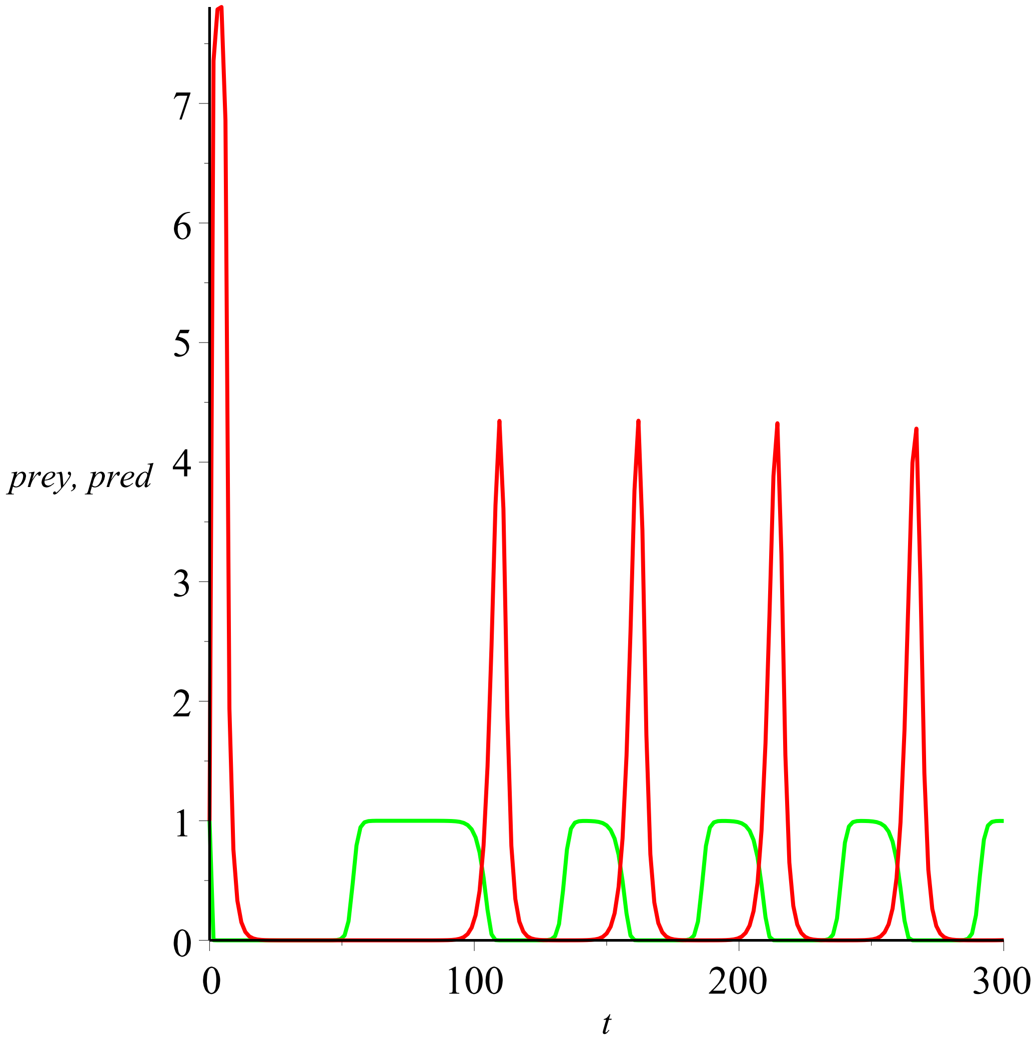
```
> dsn := dsolve(eval(ddesys, τ = 0), numeric) :
  plots[odeplot](dsn, [[t, prey(t), color = green], [t, pred(t), color = red]], 0 ..300)
```



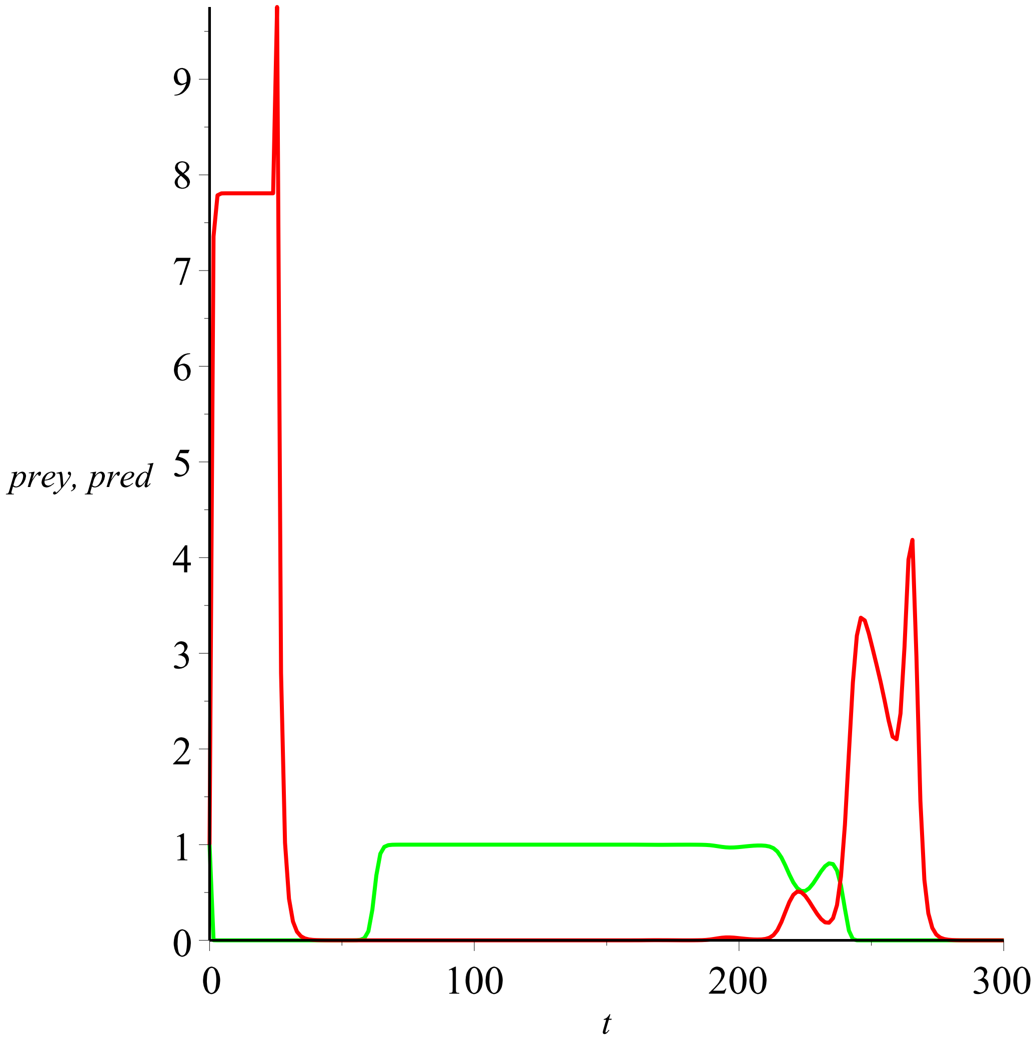

```
> dsn := dsolve(eval(ddesys, τ = 0.25), numeric) :
  plots[odeplot](dsn, [[t, prey(t), color = green], [t, pred(t), color = red]], 0 ..300)
```



> $dsn := dsolve(eval(ddesys, \tau = 5), numeric, maxfun = 0)$:
$plots[odeplot](dsn, [[t, prey(t), color = green], [t, pred(t), color = red]], 0..300)$



> $dsn := dsolve(eval(ddesys, \tau = 25), numeric)$:
$plots[odeplot](dsn, [[t, prey(t), color = green], [t, pred(t), color = red]], 0 ..300)$



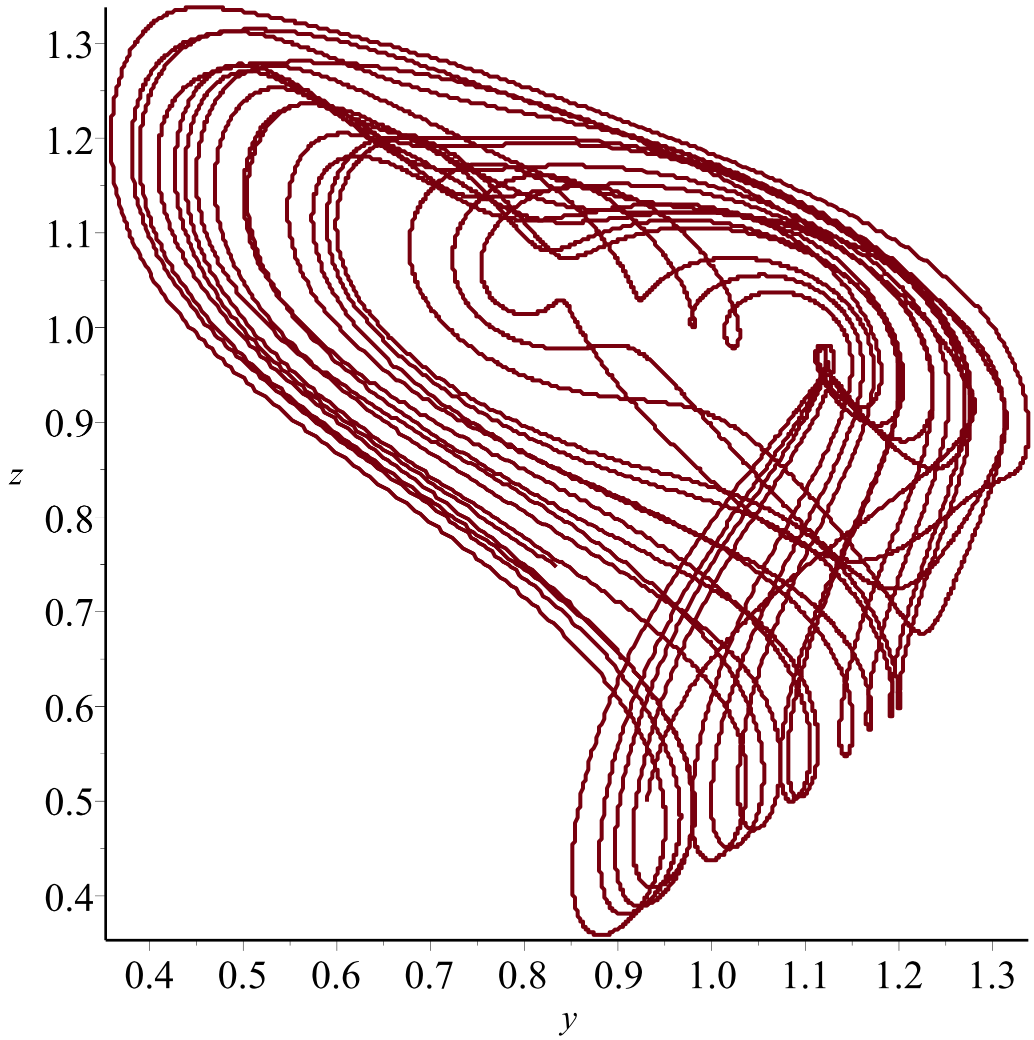
Wille and Baker Example

Demonstrates chaotic behavior for simple first order ODE with delay

> $dsn := dsolve\left(\left\{\frac{d}{dt}\, y(t) = 2 * y(t-2) / \left(1 + y(t-2)^{9.65}\right) - y(t), y(0) = 0.5, z(t) = y(t-2)\right\}, numeric\right)$

$$dsn := \mathbf{proc}(x\_rkf45\_dae) \ \ldots \ \mathbf{end\ proc} \tag{4.1}$$

> $plots[odeplot](dsn, [y(t), z(t)], 2..100, numpoints = 15000)$



Multiple Delays and History

Solve:

> $\frac{d}{dt}\, y(t) = -y(t) - 5\, y(t-1) - 2\, y(t-2)$

$$\frac{d}{dt}\, y(t) = -y(t) - 5\, y(t-1) - 2\, y(t-2) \quad \textbf{(5.1)}$$

With initial condition $y(t) = \sin(t)$ for $t < 0$

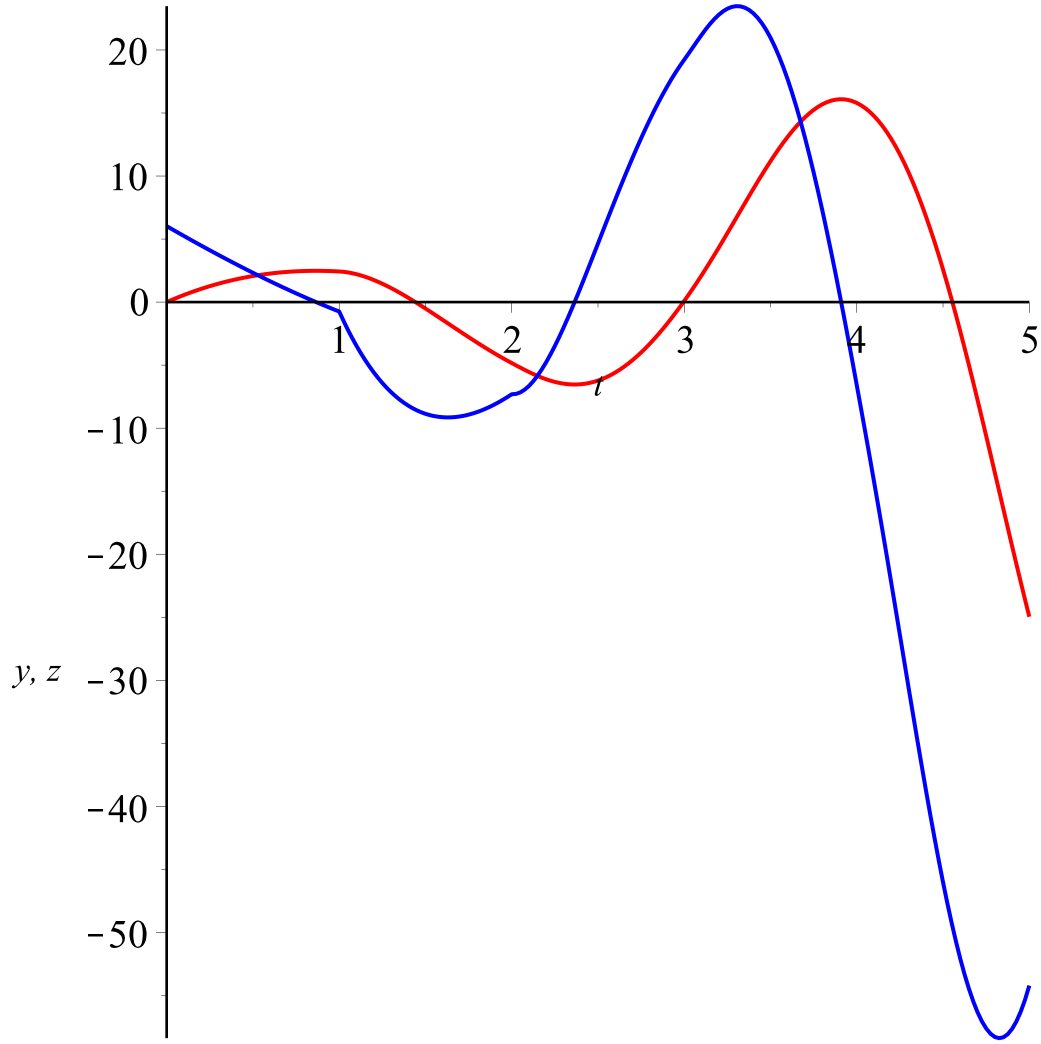
Constant assumption: use piecewise

> $ddesys := \left\{ \frac{d}{dt}\, y(t) = -y(t) - 5\, piecewise(t-1 < 0, \sin(t-1), y(t-1)) - 2\, piecewise(t-2 < 0, \sin(t-2), y(t-2)), y(0) = \sin(0), z(t) = \frac{d}{dt}\, y(t) \right\}$

$$ddesys := \left\{ \frac{d}{dt}\, y(t) = -y(t) - 5 \left(\begin{cases} \sin(t-1) & t < 1 \\ y(t-1) & otherwise \end{cases} \right) - 2 \left(\right. \quad \textbf{(5.2)}$$

$$\left\{\begin{array}{ll}\sin(t-2) & t<2\\ y(t-2) & \textit{otherwise}\end{array}\right\}, y(0)=0, z(t)=\frac{\mathrm{d}}{\mathrm{d}t}\,y(t)\Bigg\}$$

> *dsn* := *dsolve*(*ddesys*, *numeric*) :
plots[*odeplot*](*dsn*, [[*t*, *y*(*t*), *color* = *red*], [*t*, *z*(*t*), *color* = *blue*]], 0 ..5)



Suitcase Model

This is the model from the paper:

S. Suherman, R.H. Plaut, L.T. Watson, S. Thompson,
"Human delayed response time in correcting the side-to-side motion of a two wheeled suitcase.
J. Sound Vibration 207 (1997).

http://www.researchgate.net/publication/243364611_EFFECT_OF_HUMAN_RESPONSE_TIME_ON_ROCKING_INSTABILITY_OF_A_TWO-WHEELED_SUITCASE

and describes correction in the side-to-side motion of a 2-wheeled suitcase with a human delay in the response time.

The dde is as follows:

```
> dde := M d^2/dt^2 θ(t) + signum(θ(t)) M_b cos(θ(t))/2 - M_h sin(θ(t))/2 + k_0 θ(t-τ) = A sin(ω t + η)
```

$$dde := M\left(\frac{d^2}{dt^2}\theta(t)\right) + \frac{1}{2}\,\mathrm{signum}(\theta(t))\,M_b\cos(\theta(t)) - \frac{1}{2}\,M_h\sin(\theta(t)) + k_0\,\theta(t-\tau) = A\sin(\omega t + \eta) \qquad (6.1)$$

Where:

M - effective moment of inertia of suitcase rocking about either wheel
M_b - product of weight and the eff. width of the suitcase between wheels
M_h - product of weight and height of suitcase
k_0 - coefficient of the restoring moment
A, ω, η - amplitude, freq, phase of excitation moment

In addition, when the angle passes through 0, there is a loss of energy when one of the wheels impacts the ground, and this is described by a decrease in the velocity based on a coefficient of restitution, e, which we choose to have the value 0.913,

We choose the following parameter values and initial conditions:

```
> vals := {M = 1, M_b = 0.48, M_h = 1, k_0 = 1, A = 0.75, ω = 1.37, η = arcsin(M_b/A)};
  ics := θ(0) = 0, D(θ)(0) = 0
```

$$vals := \left\{A = 0.75, M = 1, M_b = 0.48, M_h = 1, \eta = \arcsin\left(\frac{M_b}{A}\right), k_0 = 1, \omega = 1.37\right\}$$

$$ics := \theta(0) = 0, D(\theta)(0) = 0 \qquad (6.2)$$

```
> ddesys := {eval(eval(dde, vals), vals), ics}
```

$$ddesys := \left\{\frac{d^2}{dt^2}\theta(t) + 0.2400000000\,\mathrm{signum}(\theta(t))\cos(\theta(t)) - \frac{1}{2}\sin(\theta(t)) + \theta(t-\tau) = 0.75\sin(1.37\,t + 0.6944982656), \theta(0) = 0, D(\theta)(0) = 0\right\} \qquad (6.3)$$

where the delay has been left unspecified.

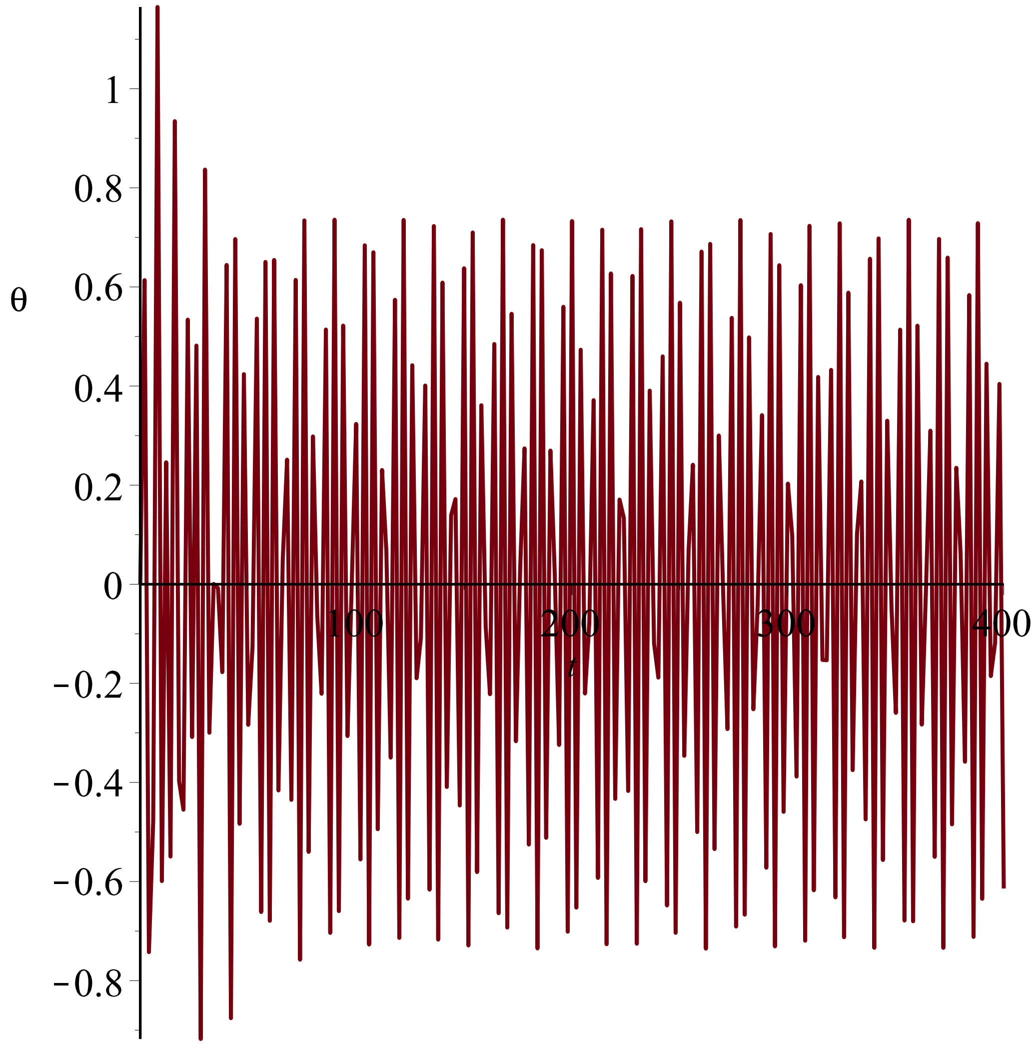
The energy loss of the wheel striking the ground is handled through the following event that states that when $\theta(t)$ passes through 0, the velocity is reduced by 0.913:

```
> evts := [[θ(t) = 0, d/dt θ(t) = 0.913 d/dt θ(t)]]
```

$$evts := \left[\left[\theta(t) = 0, \frac{d}{dt}\theta(t) = 0.913\left(\frac{d}{dt}\theta(t)\right)\right]\right] \qquad (6.4)$$

Now consider the behavior of the system if there is no delay in the response time:

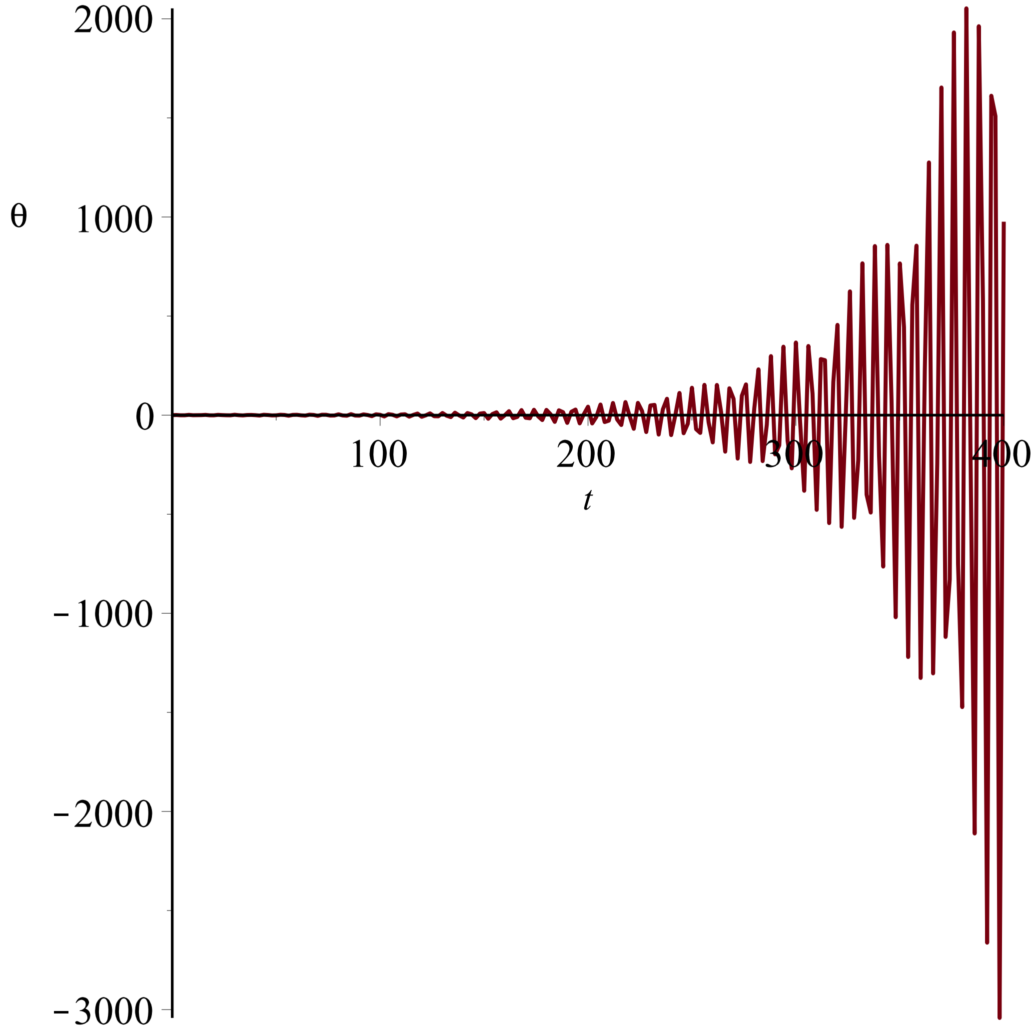
```
> dsn := dsolve(eval(ddesys, τ = 0), numeric, events = evts, maxfun = 0) :
  plots[odeplot](dsn, 0 ..400)
```



so we can see that the angle varies between approximately -0.92 and 1.16.

If, however, we introduce a 0.1 sec. delay in the response time, the situation is quite different:

> $dsn := dsolve(eval(ddesys, \tau = 0.1), numeric, events = evts, maxfun = 0)$:
> $plots[odeplot](dsn, 0..400)$



which shows that with the delay, the system is visibly unstable.