The SEIR model with births and deaths

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Background

The SEIR model is an extension of the classical SIR (Susceptibles, Infected, Recovered*)* model which was originally developed by Kermack/McKendrick[1927]: a fourth compartment is added which contains exposed persons which are infected but are not yet infectious. The SEIR (Susceptibles, Exposed, Infectious, Recovered*)* model as presented here covers also births and deaths. Persons in all compartments are measured as fractions of the total population. The birth rate is set equal to the death rate which is assumed not to be related to the infectious disease. It can be shown that the inverse of the death rate is equal to the life expectancy.

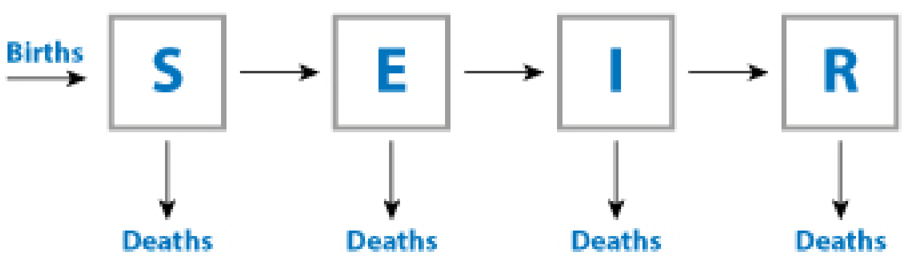


Figure 1

The SEIR model with births and deaths can be described by four differential equations with four parameters:

$S(t)$: fraction of susceptibles

$E($: fraction of exposed

$t)$ persons

$I(t)$: fraction of infectious persons

$R(t)$: fraction of recovered persons

$\mu > 0$: birth and death rate, $1/\mu$=life expectancy

$\beta > 0$: infection rate

$\sigma > 0$: exposed (infected) to infectious rate

$\gamma > 0$: removal rate

(1) $$\frac{dS(t)}{dt} = \mu - \beta \cdot S(t) \cdot I(t) - \mu \cdot S(t)$$

(2) $$\frac{dE}{dt} = \beta \cdot S(t) \cdot I(t) - \sigma \cdot E - \mu \cdot E$$

(3) $$\frac{dI(t)}{dt} = \sigma \cdot E - \gamma \cdot I(t) - \mu \cdot I(t)$$

(4) $$\frac{dR(t)}{dt} = \gamma \cdot I(t) - \mu \cdot R(t)$$

Initial conditions:

$S(0) \quad \in (0, 1]$

$E(0) \quad \in (0, 1]$

$I(0) \quad \in (0, 1]$

$R(0) \quad = 0$

$S(0) + E(0) + I(0) = 1$

The basic reproduction ratio R_0 for the SEIR model as defined above can be derived as:

$$R_0 = \frac{\beta \cdot \sigma}{(\mu + \gamma) \cdot (\mu + \sigma)}$$

R_0 is the number of secondary infections that is produced by one primary infection in a wholly susceptible population. Only when R_0 *> 1,* an epidemic occurs. *1/(γ+μ)* is the average infectious period.

Setting equations (1)-(3) equal to zero and solving these for for the equilibrium solutions of $S(t)$, $E(t)$, $I(t)$ and $R(t)$ in dependence of the parameters μ, β, σ and γ (e.g. by using the Maple procedure solve()) provides the following solution:

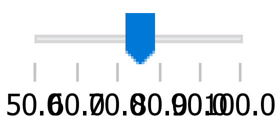
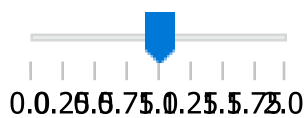
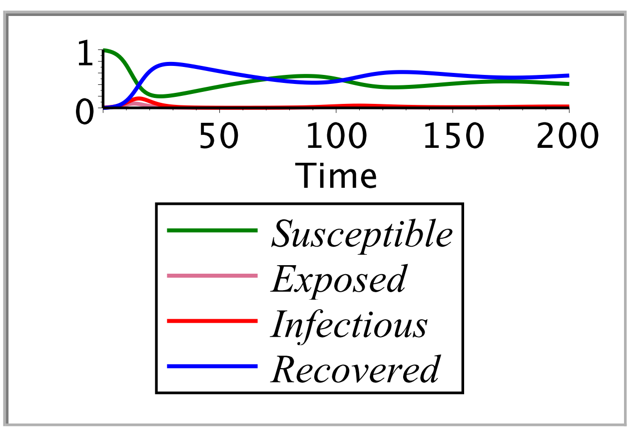
S_{eq}	=	$\frac{1}{\beta \cdot \sigma}((\mu + \gamma) \cdot (\mu + \sigma))$	=	$\frac{1}{R_0}$
E_{eq}	=	$(\mu(\beta\sigma - (\mu + \gamma) \cdot (\mu + \sigma))) / (\sigma(\mu + \sigma)\beta)$	=	$\frac{\mu \cdot (\mu + \gamma)}{\beta \cdot \sigma} \cdot (R_0 - 1)$
I_{eq}	=	$(\mu(\beta\sigma - (\mu + \gamma) \cdot (\mu + \sigma))) / (\beta \cdot (\mu + \gamma) \cdot (\mu + \sigma))$	=	$\frac{\mu}{\beta} \cdot (R_0 - 1)$
R_{eq}	=	$(\gamma \cdot (\beta\sigma - (\mu + \gamma) \cdot (\mu + \sigma))) /$	=	$1 - \frac{1}{R_0} - \frac{\mu \cdot (\mu + \gamma)}{\beta \cdot \sigma}$

		$(\beta\,(\mu+\gamma) \cdot (\mu+\sigma))$		$\cdot (R_0 - 1) - \frac{\mu}{\beta} \cdot (R_0 - 1)$

Warnings: (1) The basic reproduction ratio and the recovered persons use unfortunately the same capital letter R as symbol but one has to be aware that these are different entities. (2) Not all combinations of infection and removal rates within the ranges of the interactive model below will match a real existing infectious disease.

The SEIR model exists in different flavors and parametrizations, the model as presented here follows Keeling/Rohani[2008].

Interactive SEIR model

Life expectancy (1/μ)	Infection rate (β)
	
	
Exposed (Infected)-to-Infectious rate (σ)	Removal rate (γ)

0.0 0.25 0.5 0.75 1.0 1.25 1.5 1.75 2.0	0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8

Model summary		
Parameter name	**Symbol**	**Value**
Susceptibles, fraction at t=0	$S(0)$	0.99
Exposed (Infected), fraction at t=0	$E(0)$	0.01
Infected and infectious, fraction at t=0	$I(0)$	0.00
Recovered, fraction at t=0	$R(0)$	0.00
Birth rate=Death rate	μ	0.01
Life expectancy	$\frac{1}{\mu}$	75.00
Infection rate	β	1.00
Exposed (Infected)-to-Infectious rate	σ	1.00
Removal rate	γ	0.40
Basic reproduction ratio	$R_0 = (\beta \cdot \sigma) / ((\mu + \gamma) \cdot (\mu + \sigma))$	2.38
S_{eq}	$\frac{1}{R_0}$	0.41

E_{eq}	$\frac{\mu\cdot(\mu+\gamma)}{\beta\cdot\sigma}\cdot(R_0 - 1)$	0.007
I_{eq}	$\frac{\mu}{\beta}\cdot(R_0 - 1)$	0.018
R_{eq}	$1 - S_{eq} - E_{eq} - I_{eq}$	0.555

References

Keeling MJ, Rohani P [2008]: Modeling Infectious Diseases in Humans and Animals. Princeton University Press, Princeton and Oxford

Kermack WO, McKendrick AG [1927]: *A contribution to the Mathematical Theory of Epidemics.* Proc. Roy. Soc. A 115, 700-711 (Download link)