Descartes & Mme La Marquise du Chatelet
And The Elastic Collision of Two Bodies

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Introduction

The Marquise M^{me}du Chatelet in her book " Les Institutions Physiques" published in 1740, stated on page 36, that Descartes, when formulating his laws of motion in an elastic collision of two bodies B & C (B being more massive than C) having the same speed v, said that the smaller one C will reverse its course while the more massive body B will continue its course in the same direction as before and both will have again the same speed v.
M^{me}du Chatelet , basing her judgment on theoretical considerations using the principle of continuity , declared that Descartes was wrong in his statement. For Mme du Chatelet the larger mass B should reverse its course and move in the opposite direction. She mentioned nothing about both bodies B & C as having the same velocity after collision as Descartes did.
At the time of Descartes, some 300 years ago, the concept of kinetic energy & momentum as we know today was not yet well defined, let alone considered in any physical problem.
Actually both Descartes & M^{me}du Chatelet may have been right in some special cases but not in general as the discussion that follows will show.

Solving the Elastic Collision of Two Bodies

We consider first the general case from which we deduce the velocities after elastic collision of A & B.
We consider:
1- body A having mass = M & velocity V moving in the direction of +x-axis,
2- body B having mass = m & velocity v moving in the direction of -x-axis.
Since the collision is elastic then both **laws of conservation of kinetic energy & momentum are applicable.**
Doing so we get two equations in two unknowns:
1- $V\_$ = velocity of A after collision,
2- $v\_$ = velocity of B after collision .
We end up having two linear equations in $V\_$ & $v\_$.
When solved (see their solutions further down) we get the following result:

$$v\_ := \frac{m\,v + 2\,M\,V - M\,v}{m + M}\,, \qquad (1)$$

$$V\_ := \frac{M\,V + 2\,m\,v - m\,V}{m + M}\,, \qquad (2)$$

with the understanding that both **v & V** are **vector quantity** so that when we replace either one with its value this latter must be its **algebraic value** i.e. **negative for v** (B going in the direction of -x) & **positive for V** (A is going in the direction of +x).
Note the symmetry between the numerators of (1) & (2) : we get (2) from (1) by simply replacing v with V & m with M in the numerator.
Since in the problem stated by Descartes both A & B **have the same speed** at the time of collision and

we are working along x-axis then in the above equations we shall use V = ν & v = −ν .
Doing so equations (1) & (2) become:

$$v\_ = \frac{m(-\mathbf{v}) + 2M\mathbf{v} - M(-\mathbf{v})}{m+M} ,$$

$$v\_ = \frac{(3M - m)}{m+M}\cdot\mathbf{v} , \qquad (3)$$

$$V\_ = \frac{M\mathbf{v} + 2m(-\mathbf{v}) - m\mathbf{v}}{m+M} ,$$

$$V\_ = \frac{(M - 3m)}{m+M}\cdot\mathbf{v} . \qquad (4)$$

Case # I

If we admit the conclusion of **Descartes** then $V\_ = \frac{(M-3m)}{m+M}\cdot\mathbf{v} > 0$.

Since ν is already > 0 then we must have:

$$M - 3m > 0 \quad \rightarrow \quad M > 3m .$$

In this case body B will reverse its course and move along +x with positive velocity $v\_ = \frac{(3M - m)}{m+M}\cdot\mathbf{v}$.

But Descartes was wrong in presuming that velocity after collision is the same as before.
The following example shows all the details.

Example of *Descartes Case:* $M > 3m$

$$M = 10\ kg ;\ m = 3\ kg ;\quad \nu = 2\frac{m}{s} ;$$

Before collision:

A-velocity = + ν = V= +2, (A is going in the direction of +x),
B-velocity = - ν = v = -2 , (B is going in the direction of -x).

Kinetic Energy: $\frac{1}{2}\cdot M\cdot V^2 + \frac{1}{2}\cdot m\cdot v^2 = 26,$

Momentum: $M\cdot V + m\cdot v = 14.$

After collision:

A-velocity = $V\_ = \frac{2}{13}$, (A is going in the direction of +x),

B-velocity = $v\_ = \frac{54}{13}$, (B is going in the direction of +x).

Kinetic Energy: $\frac{1}{2}\cdot M\cdot V\_^2 + \frac{1}{2}\cdot m\cdot v\_^2 = 26,$

Momentum: $M\cdot V\_ + m\cdot v\_ = 14.$

NOTE:
It is interesting to consider the case where the mass of A is exactly 3 times the mass of B. According to equation (4), **A should stop completely** $\left(V\_ = \frac{(3m - 3m)}{m+3m}\cdot\mathbf{v} = 0\right)$ **so that all the kinetic energy of the system is invested in B whose velocity is reversed and doubled** since

$$v\_ = \frac{(3M - m)}{m+M}\cdot\mathbf{v} = \frac{(3\cdot(3m) - m)}{m+3\cdot m}\cdot\mathbf{v} = \frac{8m}{4m} = 2\mathbf{v} .$$

I don't believe we could have predicted such a result :
1- A to come to rest,
2- B to move in reverse with double its velocity.

Case # II

If we admit the conclusion of **Mme La Marquise du Chatelet** then

$$V\_ = \frac{(M - 3\,m)}{m + M} \cdot v < 0 \,.$$

Here we must have $(M - 3\,m) < 0 \;\rightarrow\; M < 3\,m$,
and since A is always larger then B, then we must have:

$$m < M < 3\,m \,.$$

Here too body B will have a positive velocity and will be going along +x.

Example of *Mme du Chatelet Case* : $m < M < 3\,m$

$$M = 8\;kg\,; \quad m = 3\;kg\,; \quad v = 2\,\frac{m}{s}\,;$$

Before collision:

A-velocity = + v = V= +2, (A is going in the direction of +x),
B-velocity = - v = v = -2, (B is going in the direction of -x).

Kinetic Energy: $\frac{1}{2}\cdot M\cdot V^2 + \frac{1}{2}\cdot m\cdot v^2$= 22,

Momentum: $M\cdot V + m\cdot v$ = 10.

After collision:

A-velocity = $V\_ = -\frac{2}{11}$, (A is going in the direction of -x),

B-velocity = $v\_ = \frac{42}{11}$, (B is going in the direction of +x).

Kinetic Energy: $\frac{1}{2}\cdot M\cdot V\_^{\,2} + \frac{1}{2}\cdot m\cdot v\_^{\,2}$= 22,

Momentum: $M\cdot V\_ + m\cdot v\_$ = 10.

Case # III

Since M is always greater than m there is no need to consider the 3d possibility of M < m.

If we consider it with $M < \frac{m}{3}$,

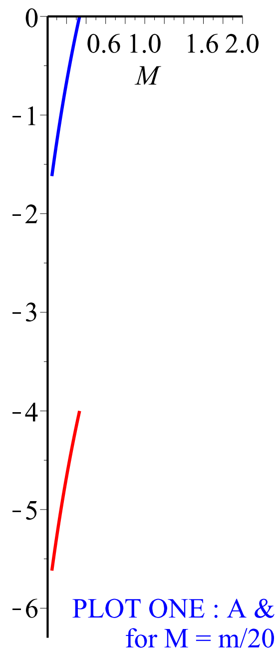
then both $V\_$ & $v\_$ are negative so that both A & B will be going along - x after collision.
This case is similar to Case # I where one of the colliding body is much larger than the other. The result is that both will move in the direction of the larger body. The larger body will be going into the direction of -x and the smaller one will follow along.

Conclusion

As we can see when both **bodies A & B have the same velocity** then their respective velocities after collision **depend on their <u>masses</u>**. This is obvious from equations (3) & (4) once plotted.

Plot of Equations (3) & (4)

In **PLOT ONE** below we have both velocities, after collision, as functions of M which varies form M= m/20 to M = m/3.
1- When M < m/3 this is the case where A-mass is << B-mass. After collision both move in the same direction (negative) till M = m/3.
2- when M = m/3, B stops moving and A, with velocity = - 4, moves in the negative direction.



PLOT ONE : A & B velocities
for M = m/20 to m/3

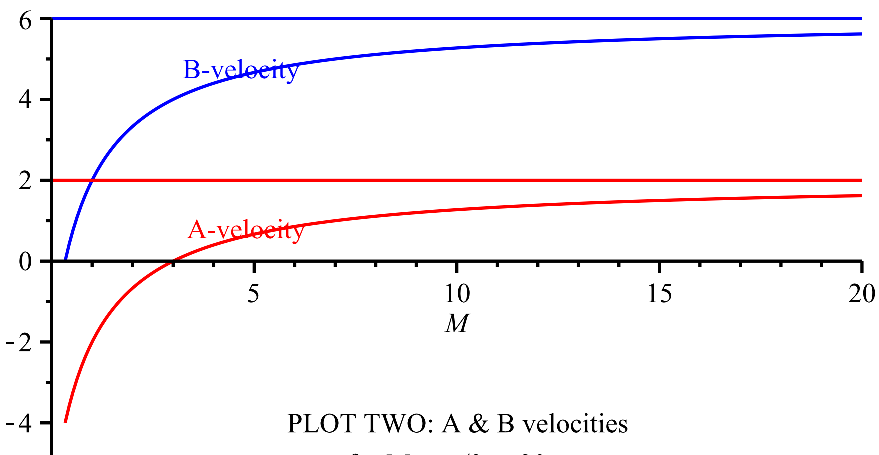
In **PLOT TWO** below we have M ranging from M = m/3 to M = 20m

3- at M = $\frac{m}{3}$, this is what we have seen above .

4- at M = m (M = 1), A & B have the same velocity of magnitude 2 with different directions.

3- at M = 3m (M = 3), A is at rest while B is moving with positive velocity = 4.

4- at M > 3m, A & B are moving in the same direction of +y.



PLOT TWO: A & B velocities
for M = m/3 to 20m

Note that the difference: $(v - V)$,

is the velocity of B as seen from A <u>right after the collision</u>.

From the conservation of the kinetic energy (see further down) we get the relation between the velocities **before** & **<u>right after the collision</u>** as

$$(v\_ - V\_) = -\ (v - V) = -\ ((-2) - 2) = 4, \qquad (5)$$

This is why the A-velocity plot & B-velocity plot after collision are identical: the B-plot is obtained from A-plot by a translation of 4 unit in the direction of y-axis.

Note also that the velocities after collision stay below their **upper limit 2 for A & 6 for B**.

From (5)

$$(v\_ - V\_ = 4),$$

which relates both velocities **<u>right after the collision</u>** we get:

$$v\_ - V\_ = 4 \ \rightarrow v\_ = V\_ + 4 = 2 + 4 = 6\,.$$

PLOT TWO shows clearly these upper limits as asymptotic lines:

y = 2 & y = 6

respectively to A & B curves.

Thus when **A is a very massive body**, <u>practically</u> the collision with B doesn't affect A velocity which remains close to 2 while B velocity is very close to 6.

This is obvious from the following example:

$$M = 100,000\ m,\ m = 1,\ \mathbf{v} = 2,$$

equations (3) & (4) give for the velocities right after collision:

$$V\_ = \frac{199994}{100001} = 1.999920001 \simeq 2 \qquad , \qquad v\_ = \frac{599998}{100001} = 5.999920001 \simeq 6,$$

while the difference is still the same:

$$v\_ - V\_ = 4 \ \rightarrow 5.999920001 - 1.999920001 = 4.$$

> *restart* :

kinetic energy before = kinetic energy after

$$> \frac{1}{2}\cdot M\cdot V^2 + \frac{1}{2}\cdot m\cdot v^2 = \frac{1}{2}\cdot M\cdot V\_^2 + \frac{1}{2}\cdot m\cdot v\_^2$$

$$\frac{1}{2}\ M\,V^2 + \frac{1}{2}\ m\,v^2 = \frac{1}{2}\ M\,V\_^2 + \frac{1}{2}\ m\,v\_^2 \qquad \textbf{(1)}$$

kinetic energy equation is put in a simple form

$$> M\cdot(V^2 - V\_^2) = m\cdot\ (v\_^2 - v^2)$$

$$M\,(V^2 - V\_^2) = m\,(v\_^2 - v^2) \qquad \textbf{(2)}$$

momentum before = momentum after

$$> M\cdot V + m\cdot v = M\cdot V\_ + m\cdot v\_$$

$$M\,V + m\,v = M\,V\_ + m\,v\_ \qquad \textbf{(3)}$$

momentum equation is put in a simple form

$$> M\cdot(V - V\_) = m\cdot\ (v\_ - v)$$

$$M\,(V - V\_) = m\,(v\_ - v) \qquad \textbf{(4)}$$

divide (2) by (4)

$$> (V + V\_) = (v + v\_)$$

$$V + V\_ = v + v\_ \qquad \textbf{(5)}$$

We end up with two linear equations (4) & (5) in two unknowns: $V\_$ & $v\_$.

From (5) we get for $V\_$:

> $V\_ = v + v\_ - V$

$$V\_ = v + v\_ - V \quad (6)$$

We put the value of $V\_$ into (4) to get equation with only $v\_$ that we solve :

> $M \cdot (V - (v + v\_ - V)) = m \cdot (v\_ - v); v\_ := solve(\%, v\_)$

$$M(2V - v - v\_) = m(v\_ - v)$$

$$v\_ := \frac{m v + 2 M V - M v}{m + M} \quad (7)$$

Putting this value of $v\_$ into (5) then solving for $V\_$:

> $V\_ := solve((V + V\_) = (v + v\_), V\_)$

$$V\_ := \frac{2 m v + M V - V m}{m + M} \quad (8)$$

<u>Here we use the values of velocities $V\_$ & $v\_$ to check that the KE is the same as before the collision.</u>

> $\frac{1}{2} \cdot M \cdot V\_^2 + \frac{1}{2} \cdot m \cdot v\_^2; simplify(\%)$

$$\frac{1}{2} \frac{M(2 m v + M V - V m)^2}{(m+M)^2} + \frac{1}{2} \frac{m(m v + 2 M V - M v)^2}{(m+M)^2}$$

$$\frac{1}{2} M V^2 + \frac{1}{2} m v^2 \quad (9)$$

<u>We check the momentum to see that it is conserved</u>

> $M \cdot V\_ + m \cdot v\_; simplify(\%)$

$$\frac{M(2 m v + M V - V m)}{m+M} + \frac{m(m v + 2 M V - M v)}{m+M}$$

$$M V + m v \quad (10)$$

It is known that the relative velocity of B (relative to A) before collision is of the same magnitude but of different direction after the collision i.e.

1- if the velocity of B as seen by A **before collision** is:

$$(v - V)$$

2- then after collision the velocity of B as seen by A becomes:

$$-(v\_ - V\_) = (V\_ - v\_) = (v - V).$$

i.e. $(v - V) = -(v\_ - V\_) \rightarrow (V + V\_) = (v + v\_).$ (I)

the algebraic sum of the velocities of A before and after the collision = algebraic sum of the velocities of B before and after the collision

Note that this relation is equivalent to the **<u>kinetic energy conservation equation.</u>**

In fact when multiplied by the momentum equation:

$$M(V - V\_) = m(v\_ - v)$$

it gives the kinetic energy equation:

$$\{(V + V\_) = (v + v\_)\} \cdot \{M(V - V\_) = m(v\_ - v)\} = M(V^2 - V\_^2) = m(v\_^2 - v^2)$$

> $-(v\_ - V\_); simplify(\%);$

$$\frac{2 m v + M V - V m}{m+M} - \frac{m v + 2 M V - M v}{m+M}$$

$$-V + v \quad (11)$$

> $restart : with(plots) :$

> $v\_ := \frac{(3M - m)}{m + M} \cdot v; \quad V\_ := \frac{(M - 3m)}{m + M} \cdot v$

$$v\_ := \frac{(3\,M - m)\,\nu}{m + M}$$

$$V\_ := \frac{(M - 3\,m)\,\nu}{m + M} \quad \textbf{(12)}$$

> $\nu := 2;\ m := 1;$

$plot\left(\,[V\_,\, v\_],\, M = \frac{m}{3}\,..5\,m,\, color = [red,\, blue],\, scaling = constrained\right) : P1 := \% :$

$plot\left(v\_ - V\_,\, M = \frac{m}{3}\,..5\,m,\, color = gold,\, scaling = constrained\right);$

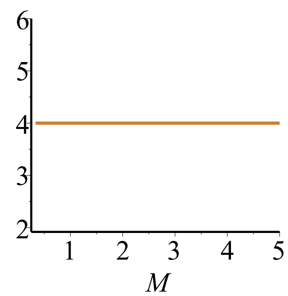
$plot(\,[2,\,6\,],\, x = 0\,..20,\, color = [red,\, blue],\, scaling = constrained) : L := \% :$

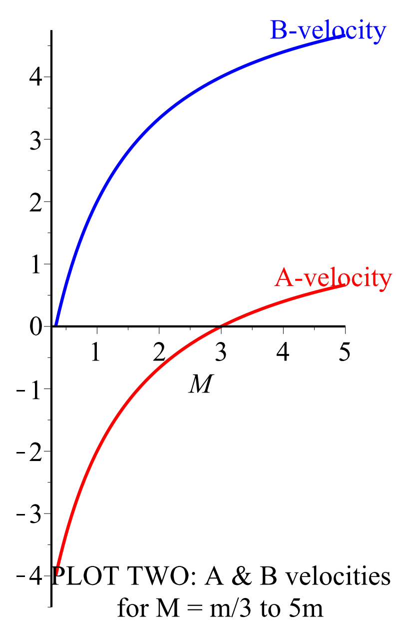
$T1 := textplot(\,[\,[4.8,\, 4.75,$ "B-velocity ", $color = blue],\, [4.8,\, 0.8,$ "A-velocity", $color = red],\, [3,\, -4,$ "PLOT TWO: A & B velocities" $],\, [3, -4.5,$ "for M = m/3 to 5m"$]\,]) :$

$display(P1,\, T1)$

$$\nu := 2$$

$$m := 1$$





> $plot\left([V\_, v\_], M = \frac{m}{20} .. \frac{m}{3}, color = [red, blue], scaling = constrained\right) : PP := \% :$

$T2 := textplot([[2, -6,$ " PLOT ONE : A & B velocities ", $color = blue], [2, -6.3,$ "for M = m/20 to m/3 ", $color = blue]]) :$

$display(PP, T2)$