Classroom Tips and Techniques: Locus of Eigenvalues

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Introduction

In a May 4, 2012, post to Maple Primes, **gdorsch** sought the locus of eigenvalues for parameter-dependent matrices. In reply, **Markiyan Hirnyk** proposed the matrix $P = M + s\,(K - M)$ as a model to study how the eigenvalues of the matrix M morph to the eigenvalues of K as the parameter s varies from 0 to 1. Apparently, the matrices M and K are real.

If the matrices M and K are not symmetric, the eigenvalues of P can become complex, making the analysis of their loci very difficult. However, on May 30, 2013, **jschulzb** appended to the earlier post the comment "I have the same problem. Have you found a good solution yet?" and the post branched to a separate new question about eigenvalue ordering. In this new post, **jschulzb** imposed symmetry on the matrices M and K, making the problem slightly more tractable.

This question about the loci of eigenvalues is reminiscent of two earlier investigations in [1] and [2]. In [1], the general question of the root locus is raised. In the context of feedback control, engineers use the locus of roots of certain polynomials to determine the stability of these feedback systems. The characteristic polynomials for the matrices $P(s)$ certainly raise similar issues.

In [2], Mike Monagan presents a Maple solution to an exercise in [3], a linear algebra text that touts numeric software as the best tool for studying that subject. In this text, Steve Leon essentially asks the student to explore the transition of the eigenvalues of one matrix to the eigenvalues of another. Because of the limited availability of access to back issues of the MapleTech journal, the next section of this article will be a paraphrase of Monagan's "reply" to Steve Leon.

Thereafter, this article will explore several examples of the loci generated by symmetric real $n \times n$ matrices $P(s) = M + s\,(K - M)$, for $n = 2$, 3, and 5. These examples will show that whether or not the eigenvalues are degenerate for some $s \in [0, 1]$, loci of eigenvalues have to be as taken curves with continuously turning tangents, that is, as C_1 curves.

Monagan's MapleTech Article

Given that the earlier editions of [3] are dated 1980, 1986, 1990, and 1994, it must be obvious that Mike Monagan's MapleTech article was based on one of the first three editions of Steve Leon's text. Mike's article used matrices that differed slightly from those in Exercise 4, page 360, in [3].

The following is a paraphrase of the exercise, and consequently, of the MapleTech article.

If $P(s) = \begin{bmatrix} 4 & -2 \\ 3 & s \end{bmatrix}$, examine the locus of its eigenvalues as s varies linearly from -5 to 5.

Solution

The eigenvalues of $P(-5)$ are $\left(-1 \pm \sqrt{57}\right)/2$ whereas the eigenvalues of $P(5)$ are $\left(9 \pm \sqrt{23}\right)/2$. Because $P(s)$ is not symmetric, its eigenvalues can be, and actually do become, complex. Indeed, the eigenvalues of $P(s)$ are $\lambda_{\pm}(s) = 2 + \left(s \pm \sqrt{s^2 - 8s - 8}\right)/2$. Figure 1 is a graph of the real values of these eigenvalues; Figure 2 shows the eigenvalues in the complex plane. In each figure, $s \in [-5, 5]$, with λ_+ drawn in black; and λ_-, in red.

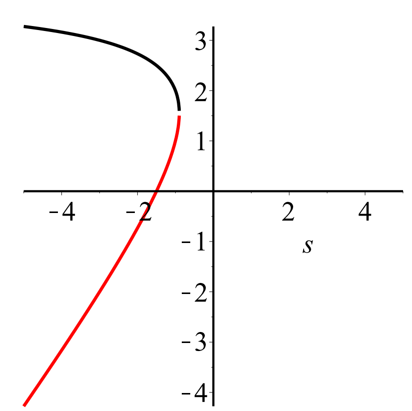


Figure 1 Eigenvalues of $P(s)$ in the real plane

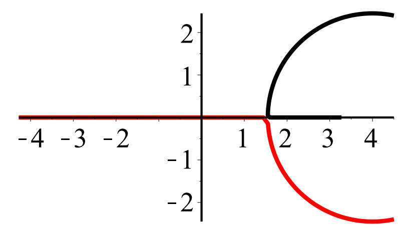


Figure 2 Eigenvalues of $P(s)$ in the complex plane

Figures 3 and 4 repeat Figures 1 and 2, respectively, but for $s \in [-5, 15]$. Figure 3 makes it clear that the analytic expressions giving $\lambda_{\pm}(s)$ are discontinuous as real functions, but not as complex-valued functions. On the other hand, Figure 4 shows that in the complex plane, the loci of $\lambda_{\pm}(s)$ do not have continuously turning tangents. Although $P(s)$ will be symmetric in the remaining examples in this article, Figures 3 and 4 portend some of the difficulties that will be encountered even for a restricted class of matrices.

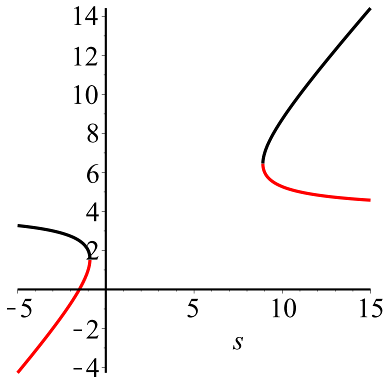


Figure 3 Eigenvalues of $P(s)$ in the real plane

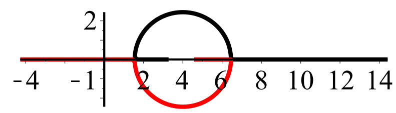


Figure 4 Eigenvalues of $P(s)$ in the complex plane

Figure 5 contains an animation of the loci traced by $\lambda_{\pm}(s)$ for $s \in [-5, 15]$. From this animation, or from an explicit calculation of the appropriate limits, the following can be deduced.

$$\lambda_{+}(s) \rightarrow \begin{cases} 4 & s \rightarrow -\infty \\ \infty & s \rightarrow \infty \end{cases} \quad \text{and}$$

$$\lambda_{-}(s) \rightarrow \begin{cases} -\infty & s \rightarrow -\infty \\ 4 & s \rightarrow \infty \end{cases}$$

The characteristic polynomial for $P(s)$ is

$$\lambda^2 + (-4 - s)\,\lambda + 6 + 4\,s$$

and its discriminant, $s^2 - 8\,s - 8$, has zeros $4 \pm 2\sqrt{6}$.

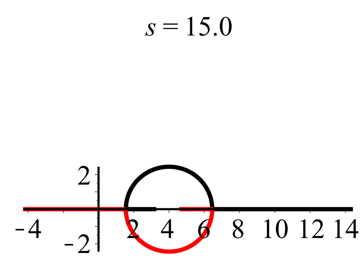


Figure 5 Animation of $\lambda_{\pm}(s)$ for $s \in [-5, 15]$

Hence, the loci of $\lambda_{\pm}$ bifurcate at $s \doteq -0.9$ and 8.9. But the real question is, do the closed-form expressions $\lambda_{\pm}(s)$ each define the locus of an eigenvalue (resulting in the black and red curves in Figure 4), or are the loci of the eigenvalues the connected curves in Figure 3?

Example 1

Find the loci of the eigenvalues of $P(s) = \begin{bmatrix} -5-2s & 6-8s \\ 6-8s & 7+s \end{bmatrix}$ for $s \in [0, 1]$.

Solution

The eigenvalues of the symmetric $P(s)$ are

$$\lambda_{\pm}(s) = 1 - \left(s \pm \sqrt{265 s^2 - 312 s + 288}\right)/2$$

Loci are graphed in Figure 6, where λ_+ is in black, and λ_- is in red.

It might seem from Figure 6 that symmetry removes many of the difficulties posed by complex eigenvalues. However, for the 2×2 matrix $P(s)$ in Example 2, $\lambda_{\pm}(1/4)$ are equal, so the loci of the eigenvalues will have a point in common. In the present example, the loci for $\lambda_{\pm}(s)$ are separate and have continuously turning tangents.

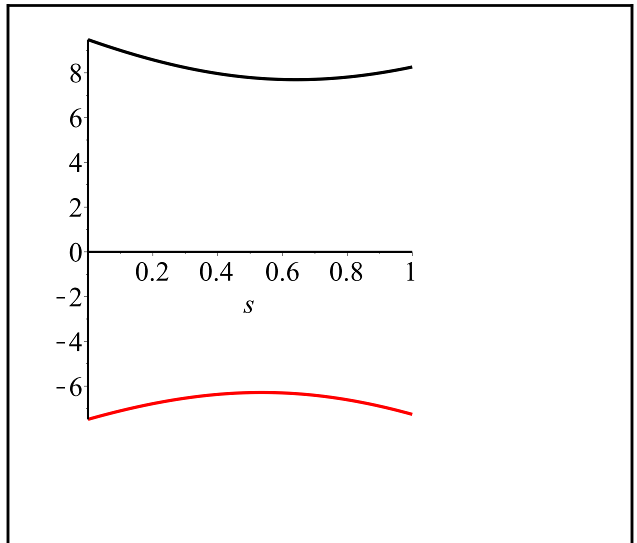


Figure 6 Loci of eigenvalues $\lambda_{\pm}(s)$, $s \in [0, 1]$

Example 2

Find the loci of the eigenvalues of $P(s) = \begin{bmatrix} 1-s & 2-8s \\ 2-8s & 3-9s \end{bmatrix}$ for $s \in [0, 1]$.

Solution

The eigenvalues of the symmetric $P(s)$ are

$$\lambda_{\pm}(s) = 2 - 5s \pm \sqrt{5}(4s - 1)$$

Loci are graphed in Figure 7, where λ_+ is in black, and λ_- is in red.

Although the loci in Figure 7 intersect, each expression for an eigenvalue generates a unique locus with continuously turning tangent. Figure 7 raises the hope that perhaps tracking an eigenvalue from $P(0)$ to $P(1)$ might be a tractable task.

But alas, even though Example 3 might reinforce this belief, Examples 4 - 6 will prove this hope to be a chimera.

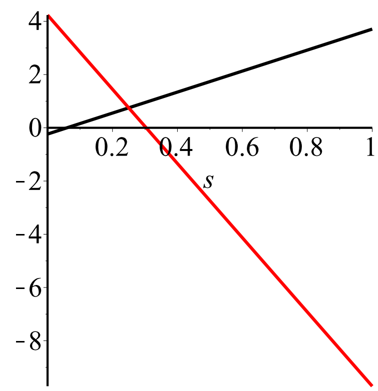


Figure 7 Intersecting loci for $\lambda_{\pm}(s)$

Example 3

Find the loci of the eigenvalues of $P(s) = \begin{bmatrix} -9+10\,s & -4+5\,s & 1-8\,s \\ -4+5\,s & 6-9\,s & -3+6\,s \\ 1-8\,s & -3+6\,s & -3+9\,s \end{bmatrix}$ for $s \in [0, 1]$.

Solution

The eigenvalues $\lambda_k(s)$, $k=1, 2, 3$, are obtained exactly with Maple's **Eigenvectors** command. The return is a list of length nearly 2000, and which would take two and a half pages to print.

Figure 8 contains a graph of the loci of the three eigenvalues, colored black, red, and green, respectively. The graph is drawn with increased precision; at standard precision, roundoff generates small imaginary parts that cause small gaps in the loci.

The loci in Figure 8 are separate and distinct, all with continuously turning tangents. For this $P(s)$ it is clearly a simple task to trace the eigenvalues from $P(0)$ to $P(1)$.

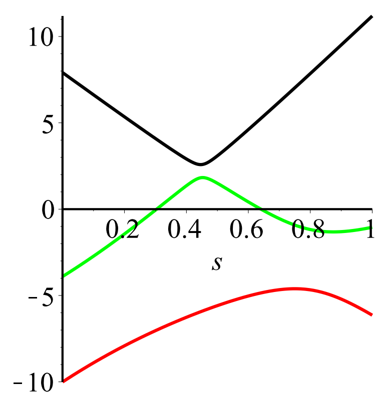


Figure 8 Loci of λ_k, $k=1, 2, 3$

Example 4

Find the loci of the eigenvalues of $P(s) = \begin{bmatrix} 3-8s & 2+16s & 4-4s \\ 2+16s & -9+20s & 8-24s \\ 4-4s & 8-24s & -5-8s \end{bmatrix}$ for $s \in [0, 1]$.

Solution

- The eigenvalues of $P(1/4)$ are $6, -8, -8$. Hence, the loci of the eigenvalues λ_2, and λ_3 have the point $(1/4, -8)$ in common. Just what this does to the loci remains to be seen.

- The characteristic equation defines $\lambda(s)$ implicitly. Maple's **implicitplot** command applied to this equation produces Figure 9 in which the curves threrefore represent the loci of the eigenvalues of $P(s)$.

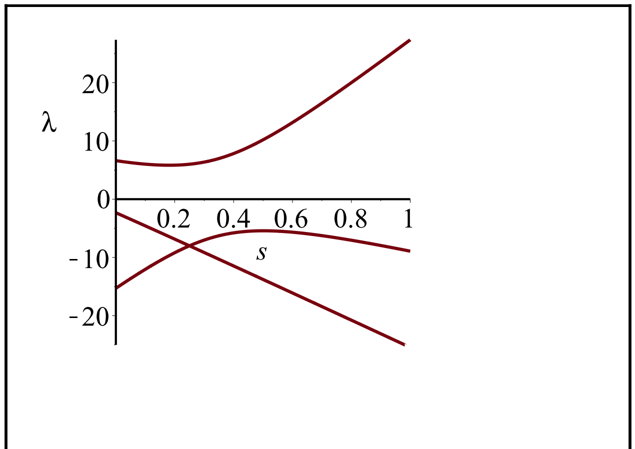


Figure 9 Loci defined implicitly by the characteristic polynomial

- In Figure 10, the loci are graphs of the exact eigenvalues obtained via Maple's **Eigenvalues** command.

- The eigenvalues of $P(0)$ are approximately $6.6, -15.3$, and -2.3 and the loci emanating from these initial points are colored black, red, and green, respectively. The eigenvalues of $P(1)$ are approximately $27.4, -8.9$, and -25.4.

- The red and green curves, each defined by an exact expression, do not have continuously turning tangents.

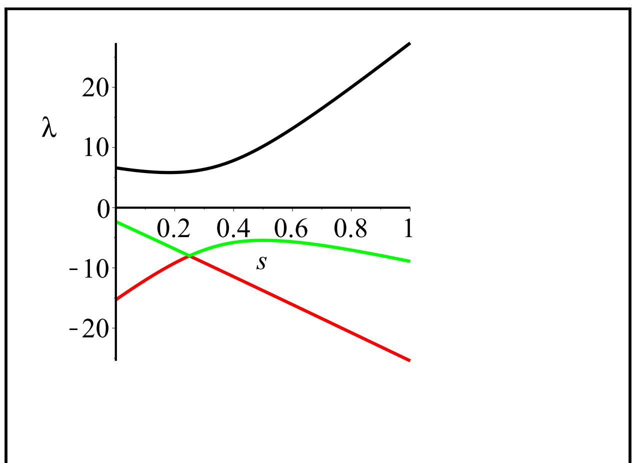


Figure 10 Loci via graph of exact expressions for the eigenvalues

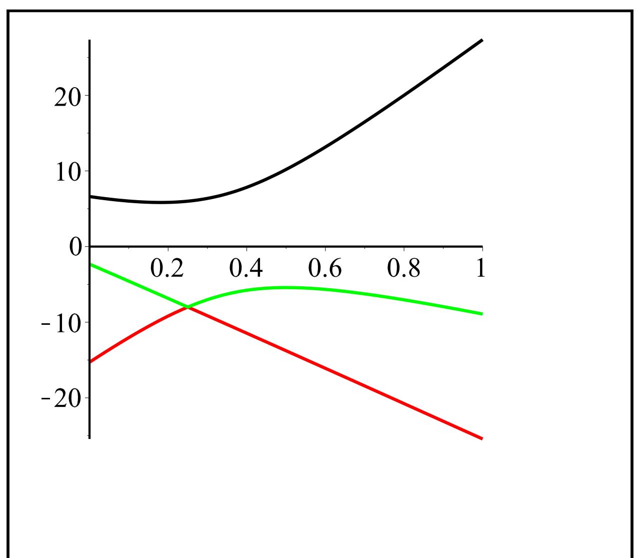
The closed-form expressions for $\lambda_k(s)$, $k = 2, 3$, are continuous, but not C_1. (This can be established analytically by the calculations in Table 1.) So either the loci of the eigenvalues are defined by the closed-form expressions $\lambda_k(s)$ and therefore do not have continuously turning tangents, or the loci are smooth curves and are only piecewise-defined by the analytic expressions $\lambda_k(s)$ whose graphs appear in Figure 10. A precise definition of the locus of eigenvalues of a real, symmetric, matrix $P(s)$ is required.

$P := \begin{bmatrix} 3-8s & 2+16s & 4-4s \\ 2+16s & -9+20s & 8-24s \\ 4-4s & 8-24s & -5-8s \end{bmatrix}:$	$\Lambda :=$ *LinearAlgebra:-Eigenvalues*$(P,$ *output* $=$ *list*$)$:
simplify$\left(\lim_{s \to 1/4-} \frac{d}{ds} \Lambda_2\right) =$ $-\frac{4}{7} + \frac{4}{7}\sqrt{1534}$ *simplify*$\left(\lim_{s \to 1/4+} \frac{d}{ds} \Lambda_2\right) =$ $-\frac{4}{7} - \frac{4}{7}\sqrt{1534}$	*simplify*$\left(\lim_{s \to 1/4-} \frac{d}{ds} \Lambda_3\right) =$ $-\frac{4}{7} - \frac{4}{7}\sqrt{1534}$ *simplify*$\left(\lim_{s \to 1/4+} \frac{d}{ds} \Lambda_3\right) =$ $-\frac{4}{7} + \frac{4}{7}\sqrt{1534}$
Table 1 Calculations showing that $\lambda_k(s)$, $k = 2, 3$, do not define C_1 curves	

Example 5

For the matrix $P(s)$ in Example 4, obtain the equivalent of Figure 10 and Table 1, using only numeric calculations.

Solution

Initializations	
• Tools>Load Package: Linear Algebra	Loading LinearAlgebra
• Tools>Load Package: Plots	Loading plots
• Define the matrix $P(s)$.	$P := \begin{bmatrix} 3-8s & 2+16s & 4-4s \\ 2+16s & -9+20s & 8-24s \\ 4-4s & 8-24s & -5-8s \end{bmatrix}:$
• Obtain the characteristic polynomial. • Define CP as a function of s.	$CP := CharacteristicPolynomial(P, \lambda):$ $F(s) = CP \xrightarrow{\text{assign as function}} F$
Numeric determination of loci of eigenvalues	
• The function G returns a list of numerically computed eigenvalues for each given value of the parameter s.	$G :=$ **proc**(t) **local** r: $r := [fsolve(F(t), \lambda, complex)]:$ **return** r: **end proc**:
• The list S contains 101 equispaced values of $s \in [0, 1]$. • Each L_j is a list of eigenvalues computed at $s = s_k$, $k = 0, \ldots, 100$.	$S := [seq(k/100, k = 0..100)]:$ **for** j **from** 1 **to** 3 **do** $L_j := [seq(G(k/100)[j], k = 0..100)]:$ **end do**:
• Constructed from the lists S and L_j, each p_j is a uniquely colored graph of the jth numerically calculated eigenvalue.	$C := [red, green, black]:$ **for** j **from** 1 **to** 3 **do** $p_j := plots\text{:-}pointplot(S, L_j, style = line, color = C_j):$ **end do**:
• Figure 11 assembles the graphs p_j, $j = 1, 2, 3$, into a single graph via Maple's **display** command; it shows that the red and green curves share the common point $(1/4, -8)$.	 **Figure 11** Numerically determined loci

Slopes on either side of $s = 1/4$ can be calculated numerically from $\lambda'(s)$ computed implicitly from the characteristic polynomial.

$$\lambda' := implicitdiff(CP, \lambda, s) = \frac{4\left(8016 s^2 + 552 s \lambda + \lambda^2 - 3096 s - 118 \lambda + 369\right)}{-1104 s^2 - 8 s \lambda + 3 \lambda^2 + 472 s + 22 \lambda - 81}$$

Recall that λ in λ' must be replaced by the appropriate eigenvalue $\lambda_k(s)$ which is available only through numeric calculation via the function *G*. The limiting process used in Table 1 can't be applied here; the requisite numeric calculations are summarized in Table 2.

$\lambda'\Big\vert_{s = 0.2499,\ \lambda = G(0.2499)[2]} =$ -22.9633224411341	$\lambda'\Big\vert_{s = 0.2499,\ \lambda = G(0.2499)[1]} =$ 21.8290002863848
$\lambda'\Big\vert_{s = 0.2501,\ \lambda = G(0.2501)[1]} =$ -22.9601959495612	$\lambda'\Big\vert_{s = 0.2501,\ \lambda = G(0.2501)[2]} =$ 21.8088009090147

Table 2 Numeric calculation of slopes along the loci on either side of $s = 1/4$

Example 6

Find the loci of the eigenvalues of

$$P(s) = \begin{bmatrix} -3 + 8s & 3 - 6s & 6 - 12s & 4 - 8s & 6 - 12s \\ 3 - 6s & 3 + 4s & 2 - 4s & -7 + 14s & -9 + 18s \\ 6 - 12s & 2 - 4s & -2 + 24s & 4 - 8s & 3 - 6s \\ 4 - 8s & -7 + 14s & 4 - 8s & 6 + 8s & 3 - 6s \\ 6 - 12s & -9 + 18s & 3 - 6s & 3 - 6s & -6 + 42s \end{bmatrix} \text{ for } s \in [0, 1].$$

Solution

Since $P(s)$ is a 5×5 matrix, only numeric techniques can be used to find its eigenvalues. However, note that the matrix has been chosen so that the eigenvalues of $P(1/2)$ are 1, 5, 10, 10, and 15.

In Table 3 the matrix *P* is defined, and the characteristic polynomial is defined as the function $F(s)$.

$$P := \begin{bmatrix} -3+8s & 3-6s & 6-12s & 4-8s & 6-12s \\ 3-6s & 3+4s & 2-4s & -7+14s & -9+18s \\ 6-12s & 2-4s & -2+24s & 4-8s & 3-6s \\ 4-8s & -7+14s & 4-8s & 6+8s & 3-6s \\ 6-12s & -9+18s & 3-6s & 3-6s & -6+42s \end{bmatrix} :$$

$CP := CharacteristicPolynomial(P, \lambda)$:

$F := unapply(CP, s)$:

Table 3 The matrix P and the characteristic polynomial as the function $F(s)$

The characteristic equation, (here, $F(s) = 0$) defines the loci of the eigenvalues of P implicitly. Figure 12 implements this insight via Maple's **implicitplot** command applied to the characteristic equation.

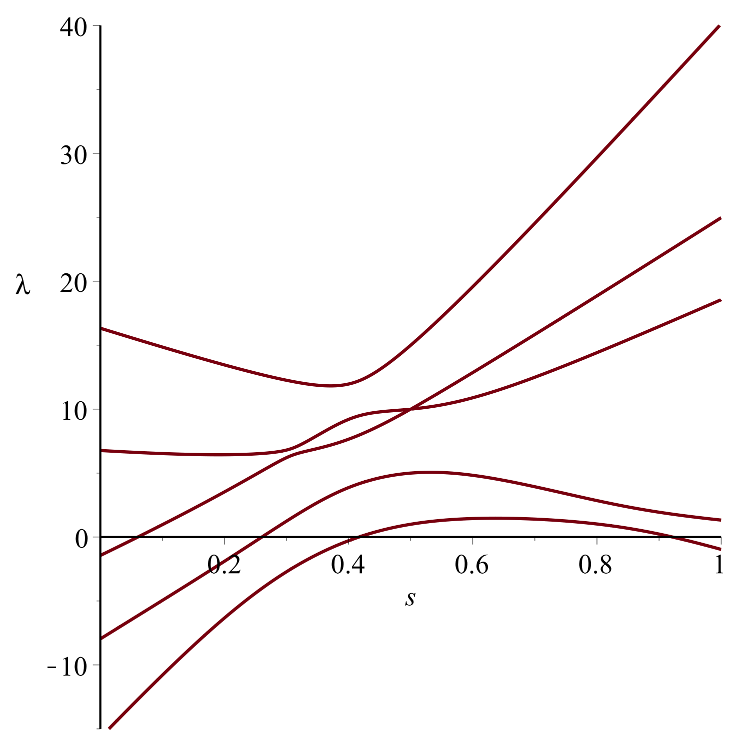


Figure 12 Loci of eigenvalues defined implicitly by the characteristic equation

Table 4 shows the calculations needed to solve for the eigenvalues numerically, and to construct the separate loci based on these numeric data.

• The function G returns a list of numerically computed eigenvalues for each given value of the parameter s.	$G := $ **proc**(t) **local** r : $r := [\mathit{fsolve}(F(t), \lambda, \mathit{complex})]$: **return** r : **end proc**:
• The list S contains 101 equispaced values of $s \in [0, 1]$. • Each L_j is a list of eigenvalues computed at $s = s_k$, $k = 0, \ldots, 100$.	$S := [\mathit{seq}(k/100, k = 0..100)]$: **for** j **from** 1 **to** 5 **do** $L_j := [\mathit{seq}(G(k/100)[j], k = 0..100)]$: **end do**:
• Constructed from the lists S and L_j, each p_j is a uniquely colored graph of the jth numerically calculated eigenvalue.	$C := [\mathit{black}, \mathit{red}, \mathit{green}, \mathit{blue}, \mathit{gold}]$: **for** j **from** 1 **to** 5 **do** $p_j := \mathit{plots{:-}pointplot}(S, L_j, \mathit{style} = \mathit{line}, \mathit{color} = C_j)$: **end do**:

Table 4 Numeric construction of the loci of eigenvalues

Figure 13 assembles the graphs p_j, $j = 1, \ldots, 5$, into a single graph via Maple's **display** command; it shows that the green and blue curves share the common point $(1/2, 10)$.

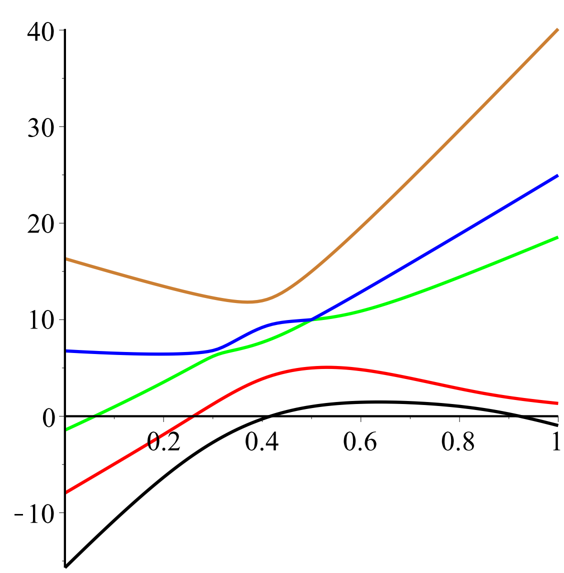


Figure 13 Numerically calculated loci of eigenvalues

For the green and blue curves in Figure 13, slopes on either side of $s = 1/2$ can be calculated numerically from $\lambda'(s)$ computed implicitly from the characteristic polynomial. Table 5 contains the relevant calculations.

• Obtain $\lambda'(s)$ implicitly with Maple's **implicitdiff** command.	$\lambda' := implicitdiff(CP, \lambda, s)$:
• Increase the number of working digits and define values of s on either side of $s = 1/2$.	$Digits := 20$: $a := .49999999$: $b := .50000001$:
Evaluate $\lambda'(s)$ on either side of $s = 1/2$ along the green and blue curves	
$\lambda'\Big\|_{s=a,\, \lambda=G(a)[4]} =$ 4.6665969631280606834 $\lambda'\Big\|_{s=b,\, \lambda=G(b)[3]} =$ 4.6665997194197342310	$\lambda'\Big\|_{s=a,\, \lambda=G(a)[3]} =$ 27.333402004025477668 $\lambda'\Big\|_{s=b,\, \lambda=G(b)[4]} =$ 27.333401289670917418
Table 5 Numeric evaluation of slopes along the green and blue curves in Figure 13	

There are no closed-form expressions for the eigenvalues of this 5 × 5 matrix $P(s)$. The eigenvalues are computed numerically by Maple's **Eigenvalues** or **fsolve** commands, each of which return a sorted list of eigenvalues. There is no user-control of this sort, but even if there were, what sorting rule could be invoked across an eigenvalue with algebraic multiplicity greater than 1? It would seem that the only way to define a unique locus of eigenvalues is to require that it be of class C_1, that is, that it have a continuously turning tangent.

References

[1]	Mathematical Thoughts on the Root Locus, Robert J. Lopez, Tips & Techniques, Maple Reporter, July, 2013.
[2]	Using Computer Algebra to Help Understand the Nature of Eigenvalues and Eigenvectors, Michael Monagan, MapleTech, Issue 9, Spring 1993, Birkhäuser.
[3]	Linear Algebra with Applications, 5th ed., Steven J. Leon, 1998, Prentice Hall.